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Third Edition

Advanced Mechanics of SOLIDS

L S Srinath

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LS Srinath received his PhD from Illinois Institute of Technology, Chicago, and has served as Professor of Mechanics and Aerospace Engineering at University of Kansas, Lawrence. He is a distinguished alumnus of IIT, Chicago. He has also served as Professor of Mechanical Engineering at the Indian Institute of Technology, Kanpur, and also the Indian Institute of Science, Bangalore. Besides these, he was the Director of Indian Institute of Technology, Madras, Chennai.

Professor Srinath has authored several books and papers and has been on Advisory Boards of quite a few national bodies dealing with engineering education.

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Chennai

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Contents

Preface

The present edition of the book is a completely revised version of the earlier two editions. The second edition provided an opportunity to correct several typographical errors and wrong answers to some problems. Also, in addition, based on many suggestions received, a chapter on composite materials was also added and this addition was well received. Since this is a second-level course addressed to senior level students, many suggestions were being received to add several specialized topics. While it was difficult to accommodate all suggestions in a book of this type, still, a few topics due to their importance needed to be included and a new edition became necessary. As in the earlier editions, the first five chapters deal with the general analysis of mechanics of deformable solids. The contents of these chapters provide a firm foundation to the mechanics of deformable solids which will enable the student to analyse and solve a variety of strength-related design problems encountered in practice. The second reason is to bring into focus the assumptions made in obtaining several basic equations. Instances are many where equations presented in handbooks are used to solve practical problems without examining whether the conditions under which those equations were obtained are satisfied or not.

The treatment starts with Analysis of stress, Analysis of strain, and Stress–Strain relations for isotropic solids. These chapters are quite exhaustive and include materials not usually found in standard books. Chapter 4 dealing with Theories of Failure or Yield Criteria is a general departure from older texts. This treatment is brought earlier because, in applying any design equation in strength related problems, an understanding of the possible factors for failure, depending on the material properties, is highly desirable. Mohr's theory of failure has been considerably enlarged because of its practical application. Chapter 5 deals with energy methods, which is one of the important topics and hence, is discussed in great detail. The discussions in this chapter are important because of their applicability to a wide variety of problems. The coverage is exhaustive and discusses the theorems of Virtual Work, Castigliano, Kirchhoff, Menabria, Engesser, and Maxwell–Mohr integrals. Several worked examples illustrate the applications of these theorems.

Bending of beams, Centre of flexure, Curved Beams, etc., are covered in Chapter 6. This chapter also discusses the validity of Euler–Bernoulli hypothesis in the derivations of beam equations. Torsion is covered in great detail in Chapter 7. Torsion of circular, elliptical, equilateral triangular bars, thin-walled multiple cell sections, etc., are discussed. Another notable inclusion in this chapter is the torsion of bars with multiply connected sections which, in spite of its importance, is not found in standard texts. Analysis of axisymmetric problems like composite tubes under internal and external pressures, rotating disks, shafts and cylinders can be found in Chapter 8.

Stresses and deformations caused in bodies due to thermal gradients need special attention because of their frequent occurrences. Usually, these problems are treated in books on Thermoelasticity. The analysis of thermal stress problems are not any more complicated than the traditional problems discussed in books on Advanced Mechanics of Solids. Chapter 9 in this book covers thermal stress problems.

Elastic instability problems are covered in Chapter 10. In addition to topics on Beam Columns, this chapter exposes the student to the instability problem as an eigenvalue problem. This is an important concept that a student has to appreciate. Energy methods as those of Rayleigh–Ritz, Timoshenko, use of trigonometric series, etc., to solve buckling problems find their place in this chapter.

Introduction to the mechanics of composites is found in Chapter 11. Modern–day engineering practices and manufacturing industries make use of a variety of composites. This chapter provides a good foundation to this topic. The subject material is a natural extension from isotropic solids to anisotropic solids. Orthotropic materials, off-axis loading, angle-ply and cross-ply laminates, failure criteria for composites, effects of Poisson's ratio, etc., are covered with adequate number of worked examples.

Stress concentration and fracture are important considerations in engineering design. Using the theory-of-elasticity approach, problems in these aspects are discussed in books solely devoted to these. However, a good introduction to these important topics can be provided in a book of the present type. Chapter 12 provides a fairly good coverage with a sufficient number of worked examples. Several practical problems can be solved with confidence based on the treatment provided.

While SI units are used in most of numerical examples and problems, a few can be found with kgf, meter and second units. This is done deliberately to make the student conversant with the use of both sets of units since in daily life, kgf is used for force and weight measurements. In those problems where kgf units are used, their equivalents in SI units are also given.

The web supplements can be accessed at http://www.mhhe.com/srinath/ams3e and it contains the following material:

For Instructors

- Solution Manual
- PowerPoint Lecture Slides

For Students

- MCQ's (interactive)
- Model Question Papers

I am thankful to all the reviewers who took out time to review this book and gave me their suggestions. Their names are given below.

K S R K Murthy	*Department of Mechanical Engineering, Indian Institute of Technology, Guwahati, Assam*
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Feedback and suggestions are always welcome at srinath_ls@sify.com.

L S Srinath

List of Symbols
(In the order they appear in the text)

σ	normal stress
F	force
$\boldsymbol{T}^{n}$	force vector on a plane with normal n
$\boldsymbol{T}^{n}_{x,y,z}$	components of force vector in x, y, z directions
A	area of section
A	normal to the section
τ	shear stress
$\sigma_{x,y,z}$	normal stress on x-plane, y-plane, z-plane
$\tau_{xy,yz,zx}$	shear stress on x-plane in y-direction, shear stress on y-plane in z-direction, shear stress on z-plane in x-direction
n_x, n_y, n_z	direction cosines of n in x, y, z directions
$\sigma_1, \sigma_2, \sigma_3$	principal stresses at a point
I_1, I_2, I_3	first, second, third invariants of stress
σ_{oct}	normal stress on octahedral plane
τ_{oct}	shear stress on octahedral plane
$\sigma_r, \sigma_\theta, \sigma_z$	normal stresses in radial, circumferential, axial (polar) direction
γ, θ, φ	spherical coordinates
$\tau_{\gamma\theta}, \tau_{\gamma z}, \tau_{\theta z}$	shear stresses in polar coordinates
u_x, u_y, u_z	displacements in x, y, z directions
E_{xx}, E_{yy}, E_{zz}	linear strains in x-direction, y-direction, z-direction (with non-linear terms)
$\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}$	linear strains (with linear terms only)
E_{xy}, E_{yz}, E_{zx}	shear strain components (with non-linear terms)
$\gamma_{xy}, \gamma_{yz}, \gamma_{zx}$	shear strain components (with linear terms only)
$\omega_x, \omega_y, \omega_z$	rigid body rotations about x, y, z axes
$\Delta = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$	cubical dilatation
$\varepsilon_1, \varepsilon_2, \varepsilon_3$	principal strains at a point
J_1, J_2, J_3	first, second, third invariants of strain

$\varepsilon_\gamma, \varepsilon_\theta, \varepsilon_z$	strains in radial, circumferential, axial directions
λ, μ	Lame's constants
$G = \mu$	rigidity modulus
μ	engineering Poisson's ratio
E	modulus of elasticity
K	bulk modulus; stress intensity factor
P	pressure
υ	Poisson's ratio
σ_y	yield point stress
U	elastic energy
U^*	distortion energy; complementary energy
σ_{ut}	ultimate stress in uniaxial tension
σ_{ct}	ultimate stress in uniaxial compression
a_{ij}	influence coefficient; material constant
b_{ij}	compliance component
M_x, M_y, M_z	moments about x, y, z axes
δ	linear deflection; generalized deflection
I_x, I_y, I_z	moments of inertia about x, y, z axes
I_ρ	polar moment of inertia
I_{xy}, I_{yz}	products of inertia about xy and yz coordinates
T	torque; temperature
Ψ	warping function
α	coefficient of thermal expansion
Q	lateral load
P	axial load
V	elastic potential
ν_{ij}	Poisson's ratio in i-direction due to stress in j-direction
b, w	width
t	thickness
K_t	theoretical stress concentration factor
N	normal force
ϕ	stream function
ρ	fillet radius
D, d	radii
q	notch sensitivity
K_c, K_{Ic}	fracture toughness in mode I
S_y	offset yield stress
ω	angular velocity
R	fracture resistance
σ_{fr}	fracture stress
Γ	boundary
J	J-integral

SI Units (Systeme International d'Unit'es)

(a) Base Units

Quantity	*Unit (Symbol)*
length	meter (m)
mass	kilogram (kg)
time	second (s)
force	newton (N)
pressure	pascal (Pa)

force is a derived unit: kgm/s^2
pressure is force per unit area: N/m^2: kg/ms^2
kilo-watt is work done per second: kNm/s

(b) Multiples

giga (G)	1 000 000 000
mega (M)	1 000 000
kilo (k)	1 000
milli (m)	0.001
micro (μ)	0.000 001
nano (n)	0.000 000 001

(c) Conversion Factors

To Convert	*to*	*Multiply by*
kgf	newton	9.8066
kgf/cm^2	Pa	9.8066×10^4
kgf/cm^2	kPa	98.066
newton	kgf	0.10197
Pa	N/m^2	1
kPa	kgf/cm^2	0.010197
HP	kW	0.746
HP	kNm/s	0.746
kW	kNm/s	1

Typical Physical Constants
(As an Aid to Solving Problems)

Material	*Ultimate Strength (MPa)*			*Yield Strength (MPa)*		*Elastic Modulus (GPa)*		*Poisson's Ratio*	*Coeff. Therm Expans. per °C*
	Tens.	*Comp*	*Shear*	*Tens or Comp*	*Shear*	*Tens*	*Shear*		$\times 10^{-6}$
Aluminium alloy	414	414	221	300	170	73	28	0.334	23.2
Cast iron, gray	210	825	—	—	—	90	41	0.211	10.4
Carbon steel	690	690	552	415	250	200	83	0.292	11.7
Stainless steel	568	568	—	276	—	207	90	0.291	17.0

For more accurate values refer to hand-books on material properties

CHAPTER 1

Analysis of Stress

1.1 INTRODUCTION

In this book we shall deal with the mechanics of deformable solids. The starting point for discussion can be either the analysis of stress or the analysis of strain. In books on the theory of elasticity, one usually starts with the analysis of strain, which deals with the geometry of deformation without considering the forces that cause the deformation. However, one is more familiar with forces, though the measurement of force is usually done through the measurement of deformations caused by the force. Books on the strength of materials, begin with the analysis of stress. The concept of stress has already been introduced in the elementry strength of materials. When a bar of uniform cross-section, say a circular rod of diameter *d*, is subjected to a tensile force *F* along the axis of the bar, the average stress induced across any transverse section perpendicular to the axis of the bar and away from the region of loading is given by

$$\sigma = \frac{F}{\text{Area}} = \frac{4F}{\pi d^2}$$

It is assumed that the reader is familiar with the elementary flexural stress and torsional stress concepts. In general, a structural member or a machine element will not possess uniform geometry of shape or size, and the loads acting on it will also be complex. For example, an automobile crankshaft or a piston inside an engine cylinder or an aircraft wing are subject to loadings that are both complex as well as dynamic in nature. In such cases, one will have to introduce the concept of the state of stress at a point and its analysis, which will be the subject of discussion in this chapter. However, we shall not deal with forces that vary with time.

It will be assumed that the matter of the body that is being considered is continuously distributed over its volume, so that if we consider a small volume element of the matter surrounding a point and shrink this volume, in the limit we shall not come across a void. In reality, however, all materials are composed of many discrete particles, which are often microscopic, and when an arbitrarily selected volume element is shrunk, in the limit one may end up in a void. But in our analysis, we assume that the matter is continuously distributed. Such a body is called a continuous medium and the mechanics of such a body or bodies is called continuum mechanics.

1.2 BODY FORCE, SURFACE FORCE AND STRESS VECTOR

Consider a body B occupying a region of space referred to a rectangular coordinate system $Oxyz$, as shown in Fig. 1.1. In general, the body will be subjected to two types of forces—body forces and surface forces. The body forces act on each volume element of the body. Examples of this kind of force are the gravitational force, the inertia force and the magnetic force. The surface forces act on the surface or area elements of the body. When the area considered lies on the actual boundary of the body, the surface force distribution is often termed surface traction. In Fig. 1.1, the surface forces F_1, F_2, F_3 . . . F_r, are concentrated forces, while p is a distributed force. The support reactions R_1, R_2 and R_3 are also surface forces. It is explicitly assumed that under the action of both body forces and surface forces, the body is in equilibrium.

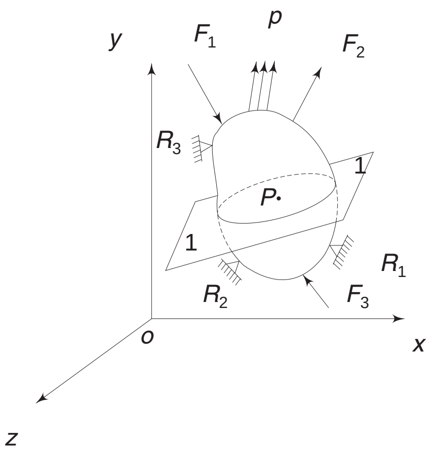

Fig. 1.1 *Body subjected to forces*

Let P be a point inside the body with coordinates (x, y, z). Let the body be cut into two parts C and D by a plane 1-1 passing through point P, as shown in Fig. 1.2. If we consider the free-body diagrams of C and D, then each part is in equilibrium under the action of the externally applied forces and the internally distributed forces across the interface. In part D, let ΔA be a small area surrounding the point P. In part C, the corresponding area at P' is $\Delta A'$. These two areas are distinguished by their outward drawn normals $\overset{1}{\boldsymbol{n}}$ and $\overset{1}{\boldsymbol{n}}'$. The action of part C on $\Delta \boldsymbol{A}$ at point P can be represented by the force vector $\Delta \overset{1}{\boldsymbol{T}}$ and the action of part D on $\Delta \boldsymbol{A}'$ at P' can be represented by the force vector $\Delta \overset{1}{\boldsymbol{T}}'$. We assume that as ΔA tends to zero, the ratio $\dfrac{\Delta \overset{1}{\boldsymbol{T}}}{\Delta \boldsymbol{A}}$ tends to a definite limit, and

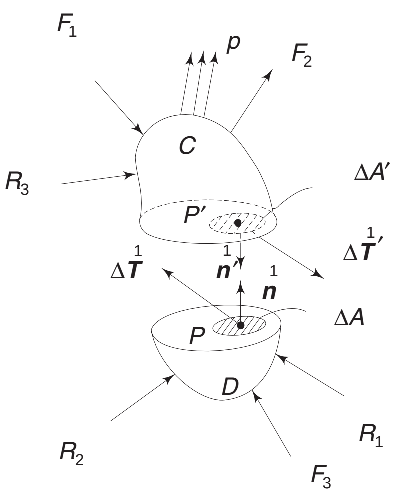

Fig. 1.2 *Free-body diagram of a body cut into two parts*

further, the moment of the forces acting on area ΔA about any point within the area vanishes in the limit. The limiting vector is written as

$$\lim_{\Delta A \to 0} \frac{\Delta \overset{1}{\boldsymbol{T}}}{\Delta A} = \frac{d\overset{1}{\boldsymbol{T}}}{dA} = \overset{1}{\boldsymbol{T}} \tag{1.1}$$

Similarly, at point P', the action of part D on C as $\Delta A'$ tends to zero, can be represented by a vector

$$0 \lim_{\Delta A' \to 0} \frac{\Delta \overset{1}{\boldsymbol{T}}'}{\Delta A'} = \frac{d\overset{1}{\boldsymbol{T}}'}{dA'} = \overset{1}{\boldsymbol{T}}' \tag{1.2}$$

Vectors $\overset{1}{\boldsymbol{T}}$ and $\overset{1}{\boldsymbol{T}}'$ are called the stress vectors and they represent forces per unit area acting respectively at P and P' on planes with outward drawn normals $\overset{1}{\boldsymbol{n}}$ and $\overset{1}{\boldsymbol{n}}'$.

We further assume that stress vector $\overset{1}{\boldsymbol{T}}$ representing the action of C on D at P is equal in magnitude and opposite in direction to stress vector $\overset{1}{\boldsymbol{T}}'$ representing the action of D on C at corresponding point P'. This assumption is similar to Newton's third law, which is applicable to particles. We thus have

$$\overset{1}{\boldsymbol{T}} = -\overset{1}{\boldsymbol{T}}' \tag{1.3}$$

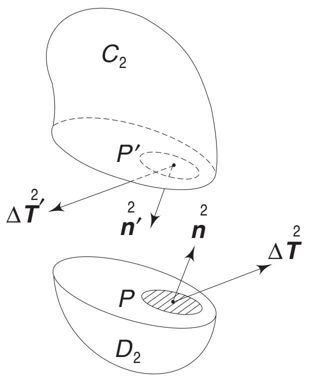

Fig. 1.3 *Body cut by another plane*

If the body in Fig. 1.1 is cut by a different plane 2-2 with outward drawn normals $\overset{2}{\boldsymbol{n}}$ and $\overset{2}{\boldsymbol{n}}'$ passing through the same point P, then the stress vector representing the action of C_2 on D_2 will be represented by $\overset{2}{\boldsymbol{T}}$ (Fig. (1.3)), i.e.

$$\overset{2}{\boldsymbol{T}} = \lim_{\Delta A \to 0} = \frac{\Delta \overset{2}{\boldsymbol{T}}}{\Delta A}$$

In general, stress vector $\overset{1}{\boldsymbol{T}}$ acting at point P on a plane with outward drawn normal $\overset{1}{\boldsymbol{n}}$ will be different from stress vector $\overset{2}{\boldsymbol{T}}$ acting at the same point P, but on a plane with outward drawn normal $\overset{2}{n}$. Hence the stress at a point depends not only on the location of the point (identified by coordinates x, y, z) but also on the plane passing through the point (identified by direction cosines n_x, n_y, n_z of the outward drawn normal).

1.3 THE STATE OF STRESS AT A POINT

Since an infinite number of planes can be drawn through a point, we get an infinite number of stress vectors acting at a given point, each stress vector characterised by the corresponding plane on which it is acting. The totality of all stress vectors acting on every possible plane passing through the point is defined to be the state of stress at the point. It is the knowledge of this state of stress that is of importance to a designer in determining the critical planes and the respective critical stresses. It will be shown in Sec. 1.6 that if the stress vectors acting on three mutually perpendicular planes passing through the point are known, we can determine the stress vector acting on any other arbitrary plane at that point.

1.4 NORMAL AND SHEAR STRESS COMPONENTS

Let $\overset{n}{\boldsymbol{T}}$ be the resultant stress vector at point P acting on a plane whose outward drawn normal is $\boldsymbol{n}$ (Fig.1.4). This can be resolved into two components, one along the normal $\boldsymbol{n}$ and the other perpendicular to $\boldsymbol{n}$. The component parallel to $\boldsymbol{n}$ is called the normal stress and is generally denoted by $\overset{n}{\sigma}$. The component perpendicular to $\boldsymbol{n}$ is known as the tangential stress or shear stress component and is denoted by $\overset{n}{\tau}$. We have, therefore, the relation:

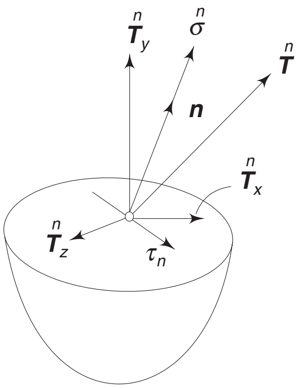

Fig. 1.4 *Resultant stress vector, normal and shear stress components*

$$\left|\overset{n}{\boldsymbol{T}}\right|^2 = \overset{n}{\sigma}{}^2 + \overset{n}{\tau}{}^2 \tag{1.4}$$

where $\left|\overset{n}{\boldsymbol{T}}\right|$ is the magnitude of the resultant stress. Stress vector $\overset{n}{\boldsymbol{T}}$ can also be resolved into three components parallel to the x, y, z axes. If these components are denoted by $\overset{n}{\boldsymbol{T}}_x, \overset{n}{\boldsymbol{T}}_y, \overset{n}{\boldsymbol{T}}_z$, we have

$$\left|\overset{n}{\boldsymbol{T}}\right|^2 = \overset{n}{\boldsymbol{T}}{}_x^2 + \overset{n}{\boldsymbol{T}}{}_y^2 + \overset{n}{\boldsymbol{T}}{}_z^2 \tag{1.5}$$

1.5 RECTANGULAR STRESS COMPONENTS

Let the body B, shown in Fig. 1.1, be cut by a plane parallel to the yz plane. The normal to this plane is parallel to the x axis and hence, the plane is called the x plane. The resultant stress vector at P acting on this will be $\overset{x}{\boldsymbol{T}}$. This vector can be resolved into three components parallel to the x, y, z axes. The component parallel to the x axis, being normal to the plane, will be denoted by σ_x (instead of by $\overset{x}{\sigma}$). The components parallel to the y and z axes are shear stress components and are denoted by τ_{xy} and τ_{xz} respectively (Fig.1.5).

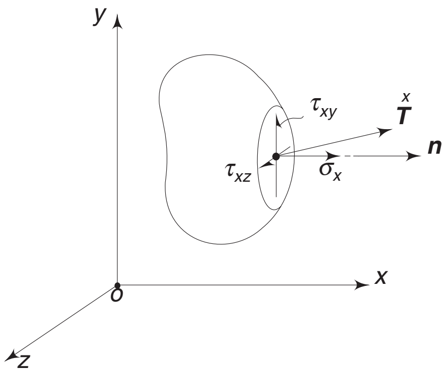

Fig. 1.5 *Stress components on x plane*

In the above designation, the first subscript x indicates the plane on which the stresses are acting and the second subscript (y or z) indicates the direction of the component. For example, τ_{xy} is the stress component on the x plane in y direction. Similarly, τ_{xz} is the stress component on the x plane in z direction. To maintain consistency, one should have denoted the normal stress component as τ_{xx}. This would be the stress component on the x plane in the x direction. However, to distinguish between a normal stress and a shear stress, the normal stress is denoted by σ and the shear stress by τ.

At any point P, one can draw three mutually perpendicular planes, the x plane, the y plane and the z plane. Following the notation mentioned above, the normal and shear stress components on these planes are

$$\sigma_x,\ \tau_{xy},\ \tau_{xz} \text{ on } x \text{ plane}$$

$$\sigma_y,\ \tau_{yx},\ \tau_{yz} \text{ on } y \text{ plane}$$

$$\sigma_z,\ \tau_{zx},\ \tau_{zy} \text{ on } z \text{ plane}$$

These components are shown acting on a small rectangular element surrounding the point P in Fig. 1.6.

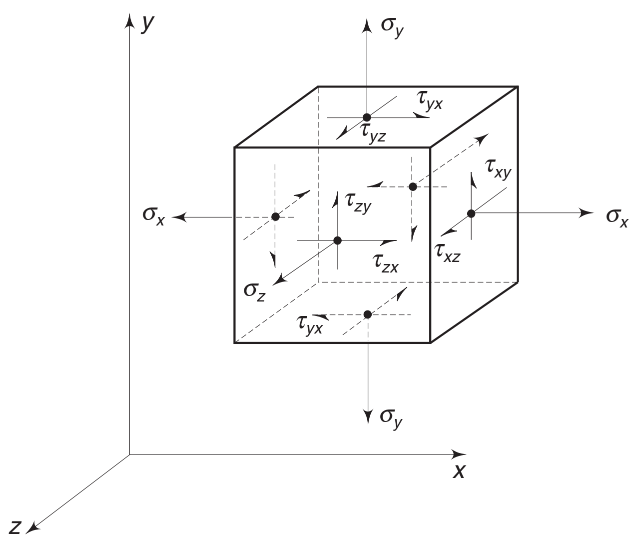

Fig. 1.6 *Rectangular stress components*

One should observe that the three visible faces of the rectangular element have their outward drawn normals along the positive x, y and z axes respectively. Consequently, the positive stress components on these faces will also be directed along the positive axes. The three hidden faces have their outward drawn normals in the negative x, y and z axes. The positive stress components on these faces will, therefore, be directed along the negative axes. For example, the bottom face has its outward drawn normal along the negative y axis. Hence, the positive stress components on this face, i.e., σ_y, τ_{yx} and τ_{yz} are directed respectively along the negative y, x and z axes.

1.6 STRESS COMPONENTS ON AN ARBITRARY PLANE

It was stated in Section 1.3 that a knowledge of stress components acting on three mutually perpendicular planes passing through a point will enable one to determine the stress components acting on any plane passing through that point. Let the three mutually perpendicular planes be the x, y and z planes and let the arbitrary plane be identified by its outward drawn normal $\boldsymbol{n}$ whose direction cosines are n_x, n_y and n_z. Consider a small tetrahedron at P with three of its faces normal to the coordinate axes, and the inclined face having its normal parallel to $\boldsymbol{n}$. Let h be the perpendicular distance from P to the inclined face. If the tetrahedron is isolated from the body and a free-body diagram is drawn, then it will be in equilibrium under the action of the surface forces and the body forces. The free-body diagram is shown in Fig. 1.7.

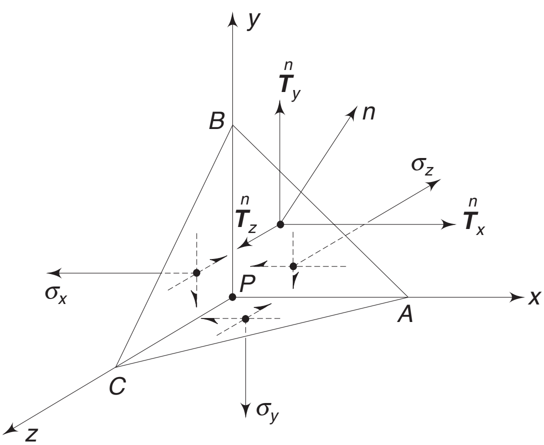

Fig. 1.7 *Tetrahedron at point P*

Since the size of the tetrahedron considered is very small and in the limit as we are going to make h tend to zero, we shall speak in terms of the average stresses over the faces. Let $\overset{n}{\boldsymbol{T}}$ be the resultant stress vector on face ABC. This can be resolved into components $\overset{n}{\boldsymbol{T}}_x, \overset{n}{\boldsymbol{T}}_y, \overset{n}{\boldsymbol{T}}_z,$ parallel to the three axes x, y and z. On the three faces, the rectangular stress components are σ_x, τ_{xy}, τ_{xz}, σ_y, τ_{yz}, τ_{yx}, σ_z, τ_{zx} and τ_{zy}. If A is the area of the inclined face then

$$\begin{aligned}
\text{Area of } BPC &= \text{projection of area } ABC \text{ on the } yz \text{ plane} \\
&= An_x \\
\text{Area of } CPA &= \text{projection of area } ABC \text{ on the } xz \text{ plane} \\
&= An_y \\
\text{Area of } APB &= \text{projection of area } ABC \text{ on the } xy \text{ plane} \\
&= An_z
\end{aligned}$$

Let the body force components in x, y and z directions be γ_x, γ_y and γ_z respectively, per unit volume. The volume of the tetrahedron is equal to $\frac{1}{3}Ah$ where h is the perpendicular distance from P to the inclined face. For equilibrium of the tetrahedron, the sum of the forces in x, y and z directions must individually vanish. Thus, for equilibrium in x direction

$$\overset{n}{\boldsymbol{T}}_x A - \sigma_x A n_x - \tau_{yx} A n_y - \tau_{zx} A n_z + \frac{1}{3} A h \gamma_x = 0$$

Cancelling A,

$$\overset{n}{\boldsymbol{T}}_x = \sigma_x n_x + \tau_{yx} n_y + \tau_{zx} n_z - \frac{1}{3} h \gamma_x \tag{1.6}$$

Similarly, for equilibrium in y and z directions

$$\overset{n}{\boldsymbol{T}}_y = \tau_{xy} n_x + \sigma_y n_y + \tau_{zy} n_z - \frac{1}{3} h \gamma_y \tag{1.7}$$

and

$$\overset{n}{\boldsymbol{T}}_z = \tau_{xz} n_x + \tau_{yz} n_y + \sigma_z n_z - \frac{1}{3} h \gamma_z \tag{1.8}$$

In the limit as h tends to zero, the oblique plane ABC will pass through point P, and the average stress components acting on the faces will tend to their respective values at point P acting on their corresponding planes. Consequently, one gets from equations (1.6)–(1.8)

$$\begin{aligned} \overset{n}{\boldsymbol{T}}_x &= n_x\,\sigma_x + n_y\,\tau_{yx} + n_z\,\tau_{zx} \\ \overset{n}{\boldsymbol{T}}_y &= n_x\,\tau_{xy} + n_y\,\sigma_y + n_z\,\tau_{zy} \\ \overset{n}{\boldsymbol{T}}_z &= n_x\,\tau_{xz} + n_y\,\tau_{yz} + n_z\,\sigma_z \end{aligned} \tag{1.9}$$

Equation (1.9) is known as Cauchy's stress formula. This equation shows that the nine rectangular stress components at P will enable one to determine the stress components on any arbitrary plane passing through point P. It will be shown in Sec. 1.8 that among these nine rectangular stress components only six are independent. This is because $\tau_{xy} = \tau_{yx}$, $\tau_{zy} = \tau_{yz}$ and $\tau_{zx} = \tau_{xz}$. This is known as the equality of cross shears. In anticipation of this result, one can write Eq. (1.9) as

$$\overset{n}{\boldsymbol{T}}_i = n_x\,\tau_{ix} + n_y\,\tau_{iy} + n_z\,\tau_{iz} = \sum_j n_j\,\tau_{ij} \tag{1.10}$$

where i and j can stand for x or y or z, and $\sigma_x = \tau_{xx}$, $\sigma_y = \tau_{yy}$ and $\sigma_z = \tau_{zz}$.

If $\overset{n}{\boldsymbol{T}}$ is the resultant stress vector on plane ABC, we have

$$\left|\overset{n}{\boldsymbol{T}}\right|^2 = \overset{n}{\boldsymbol{T}}_x^{\,2} + \overset{n}{\boldsymbol{T}}_y^{\,2} + \overset{n}{\boldsymbol{T}}_z^{\,2} \tag{1.11a}$$

If σ_n and τ_n are the normal and shear stress components, we have

$$\left|\overset{n}{\boldsymbol{T}}\right|^2 = \sigma_n^2 + \tau_n^2 \tag{1.11b}$$

Since the normal stress is equal to the projection of $\overset{n}{\boldsymbol{T}}$ along the normal, it is also equal to the sum of the projections of its components $\overset{n}{\boldsymbol{T}}_x$, $\overset{n}{\boldsymbol{T}}_y$ and $\overset{n}{\boldsymbol{T}}_z$ along $\boldsymbol{n}$. Hence,

$$\sigma_n = n_x \overset{n}{\boldsymbol{T}}_x + n_y \overset{n}{\boldsymbol{T}}_y + n_z \overset{n}{\boldsymbol{T}}_z \tag{1.12a}$$

Substituting for $\overset{n}{T}_x$, $\overset{n}{T}_y$ and $\overset{n}{T}_z$ from Eq. (1.9)

$$\sigma_n = n_x^2\sigma_x + n_y^2\sigma_y + n_z^2\sigma_z + 2n_xn_y\,\tau_{xy} + 2n_yn_z\,\tau_{yz} + 2n_zn_x\,\tau_{zx} \tag{1.12b}$$

Equation (1.11) can then be used to obtain the value of τ_n

Example 1.1 *A rectangular steel bar having a cross-section 2 cm × 3 cm is subjected to a tensile force of 6000 N (612.2 kgf). If the axes are chosen as shown in Fig. 1.8, determine the normal and shear stresses on a plane whose normal has the following direction cosines:*

(i) $n_x = n_y = \dfrac{1}{\sqrt{2}}$, $n_z = 0$

(ii) $n_x = 0$, $n_y = n_x = \dfrac{1}{\sqrt{2}}$

(iii) $n_x = n_y = n_x = \dfrac{1}{\sqrt{3}}$

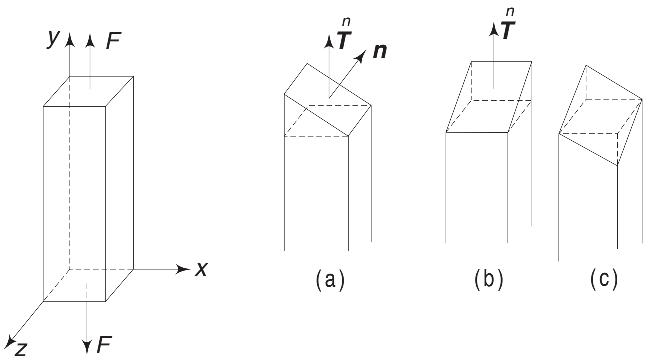

Fig. 1.8 *Example 1.1*

Solution Area of section = 2 × 3 = 6 cm^2. The average stress on this plane is 6000/6 = 1000 N/cm^2. This is the normal stress σ_y. The other stress components are zero.

(i) Using Eqs (1.9), (1.11b) and (1.12a)

$$\overset{n}{T}_x = 0, \qquad \overset{n}{T}_y = \frac{1000}{\sqrt{2}}, \qquad \overset{n}{T}_z = 0$$

$$\sigma_n = \frac{1000}{2} = 500 \text{ N/cm}^2$$

$$\tau_n^2 = \left|\overset{n}{T}\right|^2 - \sigma_n^2 = 250{,}000 \text{ N}^2/\text{cm}^4$$

$$\tau_n = 500 \text{ N/cm}^2 \ (51 \text{ kgf/cm}^2)$$

(ii) $\overset{n}{T}_x = 0, \quad \overset{n}{T}_y = \frac{1000}{\sqrt{2}}, \quad \overset{n}{T}_z = 0$

$$\sigma_n = 500 \text{ N/cm}^2, \text{ and } \tau_n = 500 \text{ N/cm}^2 \text{ (51 kgf/cm}^2)$$

(iii) $\overset{n}{T}_x = 0, \quad \overset{n}{T}_y = \frac{1000}{\sqrt{3}}, \quad \overset{n}{T}_z = 0$

$$\sigma_n = \frac{1000}{3} \text{ N/cm}^2$$

$$\tau_n = 817 \text{ N/cm}^2 \text{ (83.4 kgf/cm}^2)$$

Example 1.2 *At a point P in a body, σ_x = 10,000 N/cm² (1020 kgf/cm²), σ_y = –5,000 N/cm² (–510 kgf/cm²), σ_z = –5,000 N/cm², $\tau_{xy} = \tau_{yz} = \tau_{zx}$ = 10,000 N/cm². Determine the normal and shearing stresses on a plane that is equally inclined to all the three axes.*

Solution A plane that is equally inclined to all the three axes will have

$$n_x = n_y = n_z = \frac{1}{\sqrt{3}} \text{ since } n_x^2 + n_y^2 + n_z^2 = 1$$

From Eq. (1.12)

$$\sigma_n = \frac{1}{3}[10000 - 5000 - 5000 + 20000 + 20000 + 20000]$$

$$= 20000 \text{ N/cm}^2$$

From Eqs (1.6)–(1.8)

$$\overset{n}{T}_x = \frac{1}{\sqrt{3}}(10000 + 10000 + 10000) = 10000\sqrt{3} \text{ N/cm}^2$$

$$\overset{n}{T}_y = \frac{1}{\sqrt{3}}(10000 - 5000 + 10000) = -5000\sqrt{3} \text{ N/cm}^2$$

$$\overset{n}{T}_z = \frac{1}{\sqrt{3}}(10000 - 10000 - 5000) = -5000\sqrt{3} \text{ N/cm}^2$$

$$\therefore \quad \left|\overset{n}{T}\right|^2 = 3[(10^8) + (25 \times 10^6) + (25 \times 10^6)] \text{ N}^2/\text{cm}^4$$

$$= 450 \times 10^6 \text{ N}^2/\text{cm}^4$$

$$\therefore \quad \tau_n^2 = 450 \times 10^6 - 400 \times 10^6 = 50 \times 10^6 \text{ N}^2/\text{cm}^4$$

or $\quad \tau_n = 7000 \text{ N/cm}^2$ (approximately)

Example 1.3 *Figure 1.9 shows a cantilever beam in the form of a trapezium of uniform thickness loaded by a force P at the end. If it is assumed that the bending stress on any vertical section of the beam is distributed according to the elementary* *flexure formula, show that the normal stress σ on a section perpendicular to the top edge of the beam at point A is $\dfrac{\sigma_1}{\cos^2\theta}$, where σ_1 is the flexural stress $\dfrac{Mc}{I}$, as shown in Fig. 1.9(b).*

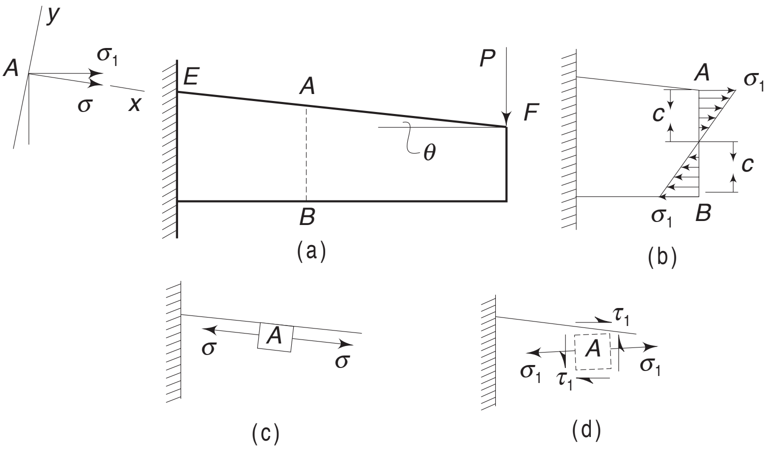

Fig. 1.9 *Example 1.3*

Solution At point A, let axes x and y be chosen along and perpendicular to the edge. On the x plane, i.e. the plane perpendicular to edge EF, the resultant stress is along the normal (i.e., x axis). There is no shear stress on this plane since the top edge is a free surface (see Sec. 1.9). But on plane AB at point A there can exist a shear stress. These are shown in Fig. 1.9(c) and (d). The normal to plane AB makes an angle θ with the x axis. Let the normal and shearing stresses on this plane be σ_1 and τ_1.

We have

$$\sigma_x = \sigma, \qquad \sigma_y = \sigma_z = 0, \qquad \tau_{xy} = \tau_{yz} = \tau_{zx} = 0$$

The direction cosines of the normal to plane AB are

$$n_x = \cos\theta, \qquad n_y = \sin\theta, \qquad n_z = 0$$

The components of the stress vector acting on plane AB are

$$\overset{n}{T}_x = \sigma_1 = n_x\,\sigma_x + n_y\,\tau_{yx} + n_z\,\tau_{zy} = \sigma\cos\theta$$

$$\overset{n}{T}_y = n_x\,\tau_{xy} + n_y\,\sigma_y + n_z\,\tau_{zy} = 0$$

$$\overset{n}{T}_z = n_x\,\tau_{xz} + n_y\,\tau_{yz} + n_z\,\sigma_z = 0$$

Therefore, the normal stress on plane $AB = \sigma_n = n_x\,\overset{n}{T}_x + n_y\,\overset{n}{T}_y + n_z\,\overset{n}{T}_z = \sigma\cos^2\theta.$

Since $\sigma_n = \sigma_1$

$$\sigma = \frac{\sigma_1}{\cos^2\theta} = \frac{Mc}{I\cos^2\theta}$$

Further, the resultant stress on plane *AB* is

$$\left|\overset{n}{\boldsymbol{T}}\right|^2 = \overset{n}{\boldsymbol{T}}{}_x^2 + \overset{n}{\boldsymbol{T}}{}_y^2 + \overset{n}{\boldsymbol{T}}{}_z^2 = \sigma^2\cos^2\theta$$

Hence $\tau^2 = \sigma^2\cos^2\theta - \sigma_n^2$

$$= \sigma^2\cos^2\theta - \sigma^2\cos^4\theta$$

or $\tau = \frac{1}{2}\sigma\sin 2\theta$

1.7 DIGRESSION ON IDEAL FLUID

By definition, an ideal fluid cannot sustain any shearing forces and the normal force on any surface is compressive in nature. This can be represented by

$$\overset{n}{\boldsymbol{T}} = -p\boldsymbol{n}, \quad p \geq 0$$

The rectangular components of $\overset{n}{\boldsymbol{T}}$ are obtained by taking the projections of $\overset{n}{\boldsymbol{T}}$ along the *x*, *y* and *z* axes. If n_x, n_y and n_z are the direction cosines of $\boldsymbol{n}$, then

$$\overset{n}{\boldsymbol{T}}_x = -pn_x, \quad \overset{n}{\boldsymbol{T}}_y = -pn_y, \quad \overset{n}{\boldsymbol{T}}_z = -pn_z \tag{1.13}$$

Since all shear stress components are zero, one has from Eqs. (1.9),

$$\overset{n}{\boldsymbol{T}}_x = n_x\sigma_x, \quad \overset{n}{\boldsymbol{T}}_y = n_y\sigma_y, \quad \overset{n}{\boldsymbol{T}}_z = n_z\sigma_z \tag{1.14}$$

Comparing Eqs (1.13) and (1.14)

$$\sigma_x = \sigma_y = \sigma_z = -p$$

Since plane $\boldsymbol{n}$ was chosen arbitrarily, one concludes that the resultant stress vector on any plane is normal and is equal to $-p$. This is the type of stress that a small sphere would experience when immersed in a liquid. Hence, the state of stress at a point where the resultant stress vector on any plane is normal to the plane and has the same magnitude is known as a hydrostatic or an isotropic state of stress. The word isotropy means ‘independent of orientation’ or ‘same in all directions’. This aspect will be discussed again in Sec. 1.14.

1.8 EQUALITY OF CROSS SHEARS

We shall now show that of the nine rectangular stress components σ_x, τ_{xy}, τ_{xz}, σ_y, τ_{yx}, τ_{yz}, σ_z, τ_{zx} and τ_{zy}, only six are independent. This is because $\tau_{xy} = \tau_{yx}$, $\tau_{yz} = \tau_{zy}$ and $\tau_{zx} = \tau_{xz}$. These are known as cross-shears. Consider an infinitesimal rectangular parallelpiped surrounding point *P*. Let the dimensions of the sides be Δx, Δy and Δz (Fig. 1.10).

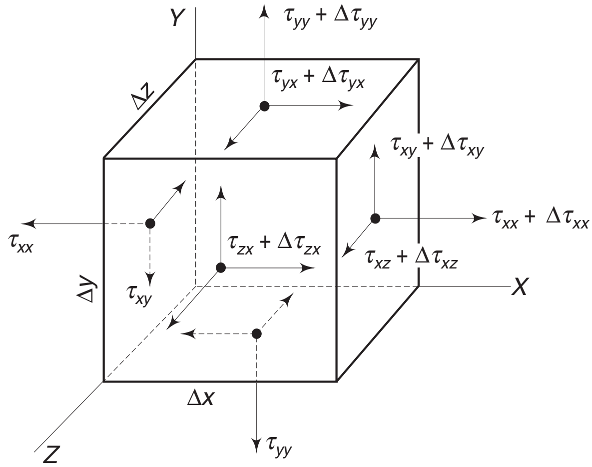

Fig. 1.10 *Stress components on a rectangular element*

Since the element considered is small, we shall speak in terms of average stresses over the faces. The stress vectors acting on the faces are shown in the figure. On the left x plane, the stress vectors are τ_{xx}, τ_{xy} and τ_{xz}. On the right face, the stresses are $\tau_{xx} + \Delta\tau_{xx}$, $\tau_{xy} + \Delta\tau_{xy}$ and $\tau_{xz} + \Delta\tau_{xz}$. These changes are because the right face is at a distance Δx from the left face. To the first order of approximation we have

$$\Delta\tau_{xx} = \frac{\partial\tau_{xx}}{\partial x}\Delta x, \qquad \Delta\tau_{xy} = \frac{\partial\tau_{xy}}{\partial x}\Delta x, \qquad \Delta\tau_{xz} = \frac{\partial\tau_{xz}}{\partial x}\Delta x$$

Similarly, the stress vectors on the top face are $\tau_{yy} + \Delta\tau_{yy}$, $\tau_{yx} + \Delta\tau_{yx}$ and $\tau_{yz} + \Delta\tau_{yz}$, where

$$\Delta\tau_{yy} = \frac{\partial\tau_{yy}}{\partial y}\Delta y, \qquad \Delta\tau_{yx} = \frac{\partial\tau_{yx}}{\partial y}\Delta y, \qquad \Delta\tau_{yz} = \frac{\partial\tau_{yz}}{\partial y}\Delta y$$

On the rear and front faces, the components of stress vectors are respectively

$$\tau_{zz}, \tau_{zx}, \tau_{zy}$$

$$\tau_{zz} + \Delta\tau_{zz}, \tau_{zx} + \Delta\tau_{zx}, \tau_{zy} + \Delta\tau_{zy}$$

where

$$\Delta\tau_{zz} = \frac{\partial\tau_{zz}}{\partial z}\Delta z, \qquad \Delta\tau_{zx} = \frac{\partial\tau_{zx}}{\partial z}\Delta z, \qquad \Delta\tau_{zy} = \frac{\partial\tau_{zy}}{\partial z}\Delta z$$

For equilibrium, the moments of the forces about the x, y and z axes must vanish individually. Taking moments about the z axis, one gets

$$\tau_{xx}\,\Delta y\,\Delta z\,\frac{\Delta y}{2} - (\tau_{xx} + \Delta\tau_{xx})\,\Delta y\,\Delta z\,\frac{\Delta y}{2} +$$

$$(\tau_{xy} + \Delta\tau_{xy})\,\Delta y\,\Delta z\,\Delta x - \tau_{yy}\,\Delta x\,\Delta z\,\frac{\Delta x}{2} +$$

$$(\tau_{yy} + \Delta\tau_{yy})\,\Delta x\,\Delta z\,\frac{\Delta x}{2} - (\tau_{yx} + \Delta\tau_{yx})\,\Delta x\,\Delta z\,\Delta y +$$

$$\tau_{zy}\,\Delta x\,\Delta y\,\frac{\Delta x}{2} - \tau_{zx}\,\Delta x\,\Delta y\,\frac{\Delta y}{2} - (\tau_{zy} + \Delta\tau_{zy})\,\Delta x\,\Delta y\,\frac{\Delta x}{2} +$$

$$(\tau_{zx} + \Delta\tau_{zx})\,\Delta x\,\Delta y\,\frac{\Delta y}{2} = 0$$

Substituting for $\Delta\tau_{xx}$, $\Delta\tau_{xy}$ etc., and dividing by $\Delta x\,\Delta y\,\Delta z$

$$-\frac{\partial\tau_{xx}}{\partial x}\frac{\Delta y}{2} + \tau_{xy} + \frac{\partial\tau_{xy}}{\partial x}\Delta x + \frac{\partial\tau_{yy}}{\partial y}\frac{\Delta y}{2} -$$

$$\tau_{yx} - \frac{\partial\tau_{yx}}{\partial y}\Delta y - \frac{\partial\tau_{zy}}{\partial z}\frac{\Delta x}{2} + \frac{\partial\tau_{zx}}{\partial z}\frac{\Delta y}{2} = 0$$

In the limit as Δx, Δy and Δz tend to zero, the above equation gives $\tau_{xy} = \tau_{yx}$. Similarly, taking moments about the other two axes, we get $\tau_{yz} = \tau_{zy}$ and $\tau_{zx} = \tau_{xz}$. Thus, the cross shears are equal, and of the nine rectangular components, only six are independent. The six independent rectangular stress components are σ_x, σ_y, σ_z, τ_{xy}, τ_{yz} and τ_{zx}.

1.9 A MORE GENERAL THEOREM

The fact that cross shears are equal can be used to prove a more general theorem which states that if $\boldsymbol{n}$ and $\boldsymbol{n}'$ define two planes (not necessarily orthogonal but in the limit passing through the same point) with corresponding stress vectors $\overset{n}{\boldsymbol{T}}$ and $\overset{n}{\boldsymbol{T}}$, then the projection of $\overset{n}{\boldsymbol{T}}$ along $\boldsymbol{n}'$ is equal to the projection of $\overset{n'}{\boldsymbol{T}}$ along $\boldsymbol{n}$, i.e. $\overset{n'}{\boldsymbol{T}} \cdot \boldsymbol{n} = \overset{n'}{\boldsymbol{T}} \cdot \boldsymbol{n}$ (see Fig. 1.11).

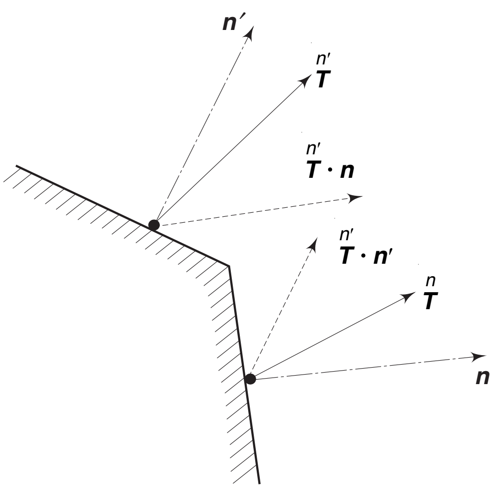

Fig. 1.11 *Planes with normals* ***n*** *and* ***n′***

The proof is straightforward. If n'_x, n'_y and n'_z are the direction cosines of $\boldsymbol{n}'$, then

$$\overset{n}{\boldsymbol{T}} \cdot \boldsymbol{n}' = \overset{n}{\boldsymbol{T}}_x n'_x + \overset{n}{\boldsymbol{T}}_y n'_y + \overset{n}{\boldsymbol{T}}_z n'_z$$

From Eq. (1.9), substituting for $\overset{n}{\boldsymbol{T}}_x$, $\overset{n}{\boldsymbol{T}}_y$ and $\overset{n}{\boldsymbol{T}}_z$ and regrouping normal and shear stresses

$$\overset{n}{\boldsymbol{T}} \cdot \boldsymbol{n}' = \sigma_x n_x n'_x + \sigma_y n_y n'_y + \sigma_z n_z n'_z + \tau_{xy} n_x n'_y + \tau_{yx} n_y n'_x +$$

$$\tau_{yz} n_y n'_z + \tau_{zy} n_z n'_y + \tau_{zx} n_z n'_x + \tau_{xz} n_x n'_z$$

Using the result $\tau_{xy} = \tau_{yx}$, $\tau_{yz} = \tau_{zy}$ and $\tau_{zx} = \tau_{zx}$

$$\overset{n}{\boldsymbol{T}} \cdot \boldsymbol{n}' = \sigma_x n_x n'_x + \sigma_y n_y n'_y + \sigma_z n_z n'_z + \tau_{xy} (n_x n'_y + n_y n'_x) + \tau_{yz} (n_y n'_z + n_z n'_y) + \tau_{zx} (n_z n'_x + n_x n'_z)$$

Similarly,

$$\overset{n'}{\boldsymbol{T}} \cdot \boldsymbol{n} = \sigma_x n_x n'_x + \sigma_y n_y n'_y + \sigma_z n_z n'_z + \tau_{xy} (n_x n'_y + n_y n'_x) + \tau_{yz} (n_y n'_z + n_z n'_y) + \tau_{zx} (n_z n'_x + n_x n'_z)$$

Comparing the above two expressions, we observe

$$\overset{n}{\boldsymbol{T}} \cdot \boldsymbol{n}' = \overset{n'}{\boldsymbol{T}} \cdot \boldsymbol{n} \tag{1.15}$$

Note: An important fact is that cross shears are equal. This can be used to prove that a shear cannot cross a free boundary. For example, consider a beam of rectangular cross-section as shown in Fig. 1.12.

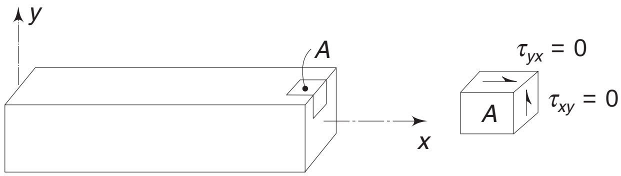

Fig. 1.12 *(a) Element with free surface; (b) Cross shears being zero*

If the top surface is a free boundary, then at point *A*, the vertical shear stres component $\tau_{xy} = 0$ because if τ_{xy} were not zero, it would call for a complementary shear τ_{yx} on the top surface. But as the top surface is an unloaded or a free surface, τ_{yx} is zero and hence, τ_{xy} is also zero (refer Example 1.3).

1.10 PRINCIPAL STRESSES

We have seen that the normal and shear stress components can be determined on any plane with normal ***n***, using Cauchy's formula given by Eqs (1.9). From the strength or failure considerations of materials, answers to the following questions are important:

(i) Are there any planes passing through the given point on which the resultant stresses are wholly normal (in other words, the resultant stress vector is along the normal)?

(ii) What is the plane on which the normal stress is a maximum and what is its magnitude?

(iii) What is the plane on which the tangential or shear stress is a maximum and what it is its magnitude?

Answers to these questions are very important in the analysis of stress, and the next few sections will deal with these. Let us assume that there is a plane ***n*** with direction cosines n_x, n_y and n_z on which the stress is wholly normal. Let σ be the magnitude of this stress vector. Then we have

$$\overset{n}{\boldsymbol{T}} = \sigma \boldsymbol{n} \tag{1.16}$$

The components of this along the x, y and z axes are

$$\overset{n}{\boldsymbol{T}}_x = \sigma n_x, \qquad \overset{n}{\boldsymbol{T}}_y = \sigma n_y, \qquad \overset{n}{\boldsymbol{T}}_z = \sigma n_z \tag{1.17}$$

Also, from Cauchy's formula, i.e. Eqs (1.9),

$$\overset{n}{\boldsymbol{T}}_x = \sigma_x n_x + \tau_{xy} n_y + \tau_{xz} n_z$$

$$\overset{n}{\boldsymbol{T}}_y = \tau_{xy} n_x + \sigma_y n_y + \tau_{yz} n_z$$

$$\overset{n}{\boldsymbol{T}}_z = \tau_{xz} n_x + \tau_{yz} n_y + \sigma_z n_z$$

Subtracting Eq. (1.17) from the above set of equations we get

$$\begin{aligned}(\sigma_x - \sigma) n_x + \tau_{xy} n_y + \tau_{xz} n_z &= 0\\ \tau_{xy} n_x + (\sigma_y - \sigma) n_y + \tau_{yz} n_z &= 0\\ \tau_{xz} n_x + \tau_{yz} n_y + (\sigma_z - \sigma) n_z &= 0\end{aligned} \tag{1.18}$$

We can view the above set of equations as three simultaneous equations involving the unknowns n_x, n_y and n_z. These direction cosines define the plane on which the resultant stress is wholly normal. Equation (1.18) is a set of homogeneous equations. The trivial solution is $n_x = n_y = n_z = 0$. For the existence of a non-trivial solution, the determinant of the coefficients of n_x, n_y and n_z must be equal to zero, i.e.

$$\begin{vmatrix} (\sigma_x - \sigma) & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & (\sigma_y - \sigma) & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & (\sigma_z - \sigma) \end{vmatrix} = 0 \tag{1.19}$$

Expanding the above determinant, one gets a cubic equation in σ as

$$\sigma^3 - (\sigma_x + \sigma_y + \sigma_z)\sigma^2 + (\sigma_x \sigma_y + \sigma_y \sigma_z + \sigma_z \sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2)\,\sigma -$$

$$(\sigma_x \sigma_y \sigma_z + 2\tau_{xy} \tau_{yz} \tau_{zx} - \sigma_x \tau_{yz}^2 - \sigma_y \tau_{xz}^2 - \sigma_z \tau_{xy}^2) = 0 \tag{1.20}$$

The three roots of the cubic equation can be designated as σ_1, σ_2 and σ_3. It will be shown subsequently that all these three roots are real. We shall later give a method (Example 4) to solve the above cubic equation. Substituting any one of these three solutions in Eqs (1.18), we can solve for the corresponding n_x, n_y and n_z. In order to avoid the trivial solution, the condition.

$$n_x^2 + n_y^2 + n_z^2 = 1 \tag{1.21}$$

is used along with any two equations from the set of Eqs (1.18). Hence, with each σ there will be an associated plane. These planes on each of which the stress vector is wholly normal are called the principal planes, and the corresponding stresses, the principal stresses. Since the resultant stress is along the normal, the tangential stress component on a principal plane is zero, and consequently, the principal plane is also known as the shearless plane. The normal to a principal plane is called the principal stress axis.

1.11 STRESS INVARIANTS

The coefficients of σ^2, σ and the last term in the cubic Eq. (1.20) can be written as follows:

$$l_1 = \sigma_x + \sigma_y + \sigma_z \tag{1.22}$$

$$l_2 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2$$

$$= \begin{vmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{vmatrix} + \begin{vmatrix} \sigma_y & \tau_{yz} \\ \tau_{yz} & \sigma_z \end{vmatrix} + \begin{vmatrix} \sigma_x & \tau_{xz} \\ \tau_{xz} & \sigma_z \end{vmatrix} \tag{1.23}$$

$$l_3 = \sigma_x\, \sigma_y\, \sigma_z + 2\tau_{xy}\, \tau_{yz}\, \tau_{zx} - \sigma_x\, \tau_{yz}^2 - \sigma_y\, \tau_{zx}^2 - \sigma_z\, \tau_{xy}^2$$

$$= \begin{vmatrix} \sigma_x & \tau_{xy} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{yz} & \sigma_z \end{vmatrix} \tag{1.24}$$

Equation (1.20) can then be written as

$$\sigma^3 - l_1\sigma^2 + l_2\sigma - l_3 = 0$$

The quantities l_1, l_2 and l_3 are known as the first, second and third invariants of stress respectively. An invariant is one whose value does not change when the frame of reference is changed. In other words if x', y', z', is another frame of reference at the same point and with respect to this frame of reference, the rectangular stress competence are $\sigma_{x'}, \sigma_{y'}, \sigma_{z'}, \tau_{x'y'}, \tau_{y'z'}$ and $\tau_{z'x'}$, then the values of l_1, l_2 and l_3, calculated as in Eqs (1.22) – (1.24), will show that

$$\sigma_x + \sigma_y + \sigma_z = \sigma_x' + \sigma_y' + \sigma_z'$$

i.e. $$l_1 = l_1'$$

and similarly, $$l_2 = l_2' \quad \text{and} \quad l_3 = l_3'$$

The reason for this can be explained as follows. The principal stresses at a point depend only on the state of stress at that point and not on the frame of reference describing the rectangular stress components. Hence, if xyz and $x'y'z'$ are two orthogonal frames of reference at the point, then the following cubic equations

$$\sigma^3 - l_1\,\sigma^2 + l_2\,\sigma - l_3 = 0$$

and $$\sigma^3 - l'_1\,\sigma^2 + l'_2\,\sigma - l'_3 = 0$$

must give the same solutions for σ. Since the two systems of axes were arbitrary, the coefficients of σ^2, and σ and the constant terms in the two equations must be equal, i.e.

$$l_1 = l_1', \qquad l_2 = l_2' \qquad \text{and} \qquad l_3 = l_3'$$

In terms of the principal stresses, the invariants are

$$l_1 = \sigma_1 + \sigma_2 + \sigma_3$$
$$l_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1$$
$$l_3 = \sigma_1\sigma_2\sigma_3$$

1.12 PRINCIPAL PLANES ARE ORTHOGONAL

The principal planes corresponding to a given state of stress at a point can be shown to be mutually orthogonal. To prove this, we make use of the general theorem in Sec. 1.9. Let $\boldsymbol{n}$ and $\boldsymbol{n}'$ be the two principal planes and σ_1 and σ_2, the corresponding principal stresses. Then the projection of σ_1 in direction $\boldsymbol{n}'$ is equal to the projection of σ_2 in direction $\boldsymbol{n}$, i.e.

$$\sigma_1 \boldsymbol{n}' \cdot \boldsymbol{n} = \sigma_2 \boldsymbol{n} \cdot \boldsymbol{n}' \tag{1.25}$$

If n_x, n_y and n_z are the direction cosines of $\boldsymbol{n}$, and n'_x, n'_y and n'_z those of $\boldsymbol{n}'$, then expanding Eq. (1.25)

$$\sigma_1\left(n_x n_x' + n_y n_y' + n_z n_z'\right) = \sigma_2\left(n_x n_x' + n_y n_y' + n_z n_z'\right)$$

Since in general, σ_1 and σ_2 are not equal, the only way the above equation can hold is

$$n_x n_x' + n_y n_y' + n_z n_z' = 0$$

i.e. $\boldsymbol{n}$ and $\boldsymbol{n}'$ are perpendicular to each other. Similarly, considering two other planes $\boldsymbol{n}'$ and $\boldsymbol{n}''$ on which the principal stresses σ_2 and σ_3 are acting, and following the same argument as above, one finds that $\boldsymbol{n}'$ and $\boldsymbol{n}''$ are perpendicular to each other. Similarly, $\boldsymbol{n}$ and $\boldsymbol{n}''$ are perpendicular to each other. Consequently, the principal planes are mutually perpendicular.

1.13 CUBIC EQUATION HAS THREE REAL ROOTS

In Sec. 1.10, it was stated that Eq. (1.20) has three real roots. The proof is as follows. Dividing Eq. (1.20) by σ^2,

$$\sigma - l_1 + \frac{l_2}{\sigma} - \frac{l_3}{\sigma^2} = 0$$

For appropriate values of σ, the quantity on the left-hand side will be equal to zero. For other values, the quantity will not be equal to zero and one can write the above function as

$$\sigma - l_1 + \frac{l_2}{\sigma} - \frac{l_3}{\sigma^2} = f(\sigma) \tag{1.26}$$

Since l_1, l_2 and l_3 are finite, $f(\sigma)$ can be made positive for large positive values of σ. Similarly, $f(\sigma)$ can be made negative for large negative values of σ. Hence, if one

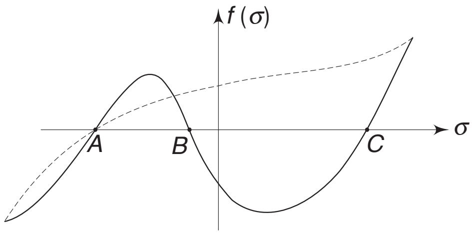

Fig.1.13 *Plot of f(σ) versus σ*

plots $f(\sigma)$ for different values of σ as shown in Fig. 1.13, the curve must cut the σ axis at least once as shown by the dotted curve and for this value of σ, $f(\sigma)$ will be equal to zero. Therefore, there is at least one real root.

Let σ_3 be this root and $\boldsymbol{n}$ the associated plane. Since the state of stress at the point can be characterised by the six rectangular components referred to any orthogonal frame of reference, let us choose a particular one, $x'y'z'$, where the z' axis is along $\boldsymbol{n}$ and the other two axes, x' and y', are arbitrary. With reference to this system, the stress matrix has the form.

$$\begin{bmatrix} \sigma_{x'} & \tau_{x'y'} & 0 \\ \tau_{x'y'} & \sigma_{y'} & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \tag{1.27}$$

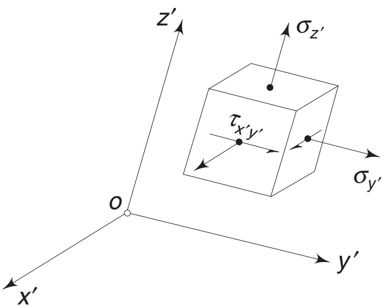

Fig 1.14 *Rectangular element with faces normal to x′, y′, z′ axes*

Figure 1.14 shows these stress vectors on a rectangular element. The shear stress components $\tau_{x'z'}$ and $\tau_{y'z'}$ are zero since the z' plane is chosen to be the principal plane. With reference to this system, Eq. (1.19) becomes

$$\begin{vmatrix} (\sigma_{x'} - \sigma) & \tau_{x'y'} & 0 \\ \tau_{x'y'} & (\sigma_{y'} - \sigma) & 0 \\ 0 & 0 & (\sigma_3 - \sigma) \end{vmatrix} = 0 \tag{1.28}$$

Expanding $\quad (\sigma_3 - \sigma)\,[\sigma^2 - (\sigma_{x'} + \sigma_{y'})\,\sigma + \sigma_{x'}\sigma_{y'} - \tau^2_{x'y'}] = 0$

This is a cubic in σ. One of the solutions is $\sigma = \sigma_3$. The two other solutions are obtained by solving the quadratic inside the brakets. The two solutions are

$$\sigma_{1,2} = \frac{\sigma_{x'} + \sigma_{y'}}{2} \pm \left[\left(\frac{\sigma_{x'} - \sigma_{y'}}{2}\right)^2 + \tau^2_{x'y'}\right]^{\frac{1}{2}} \tag{1.29}$$

The quantity under the square root $\left(\text{power } \frac{1}{2}\right)$ is never negative and hence, σ_1 and σ_2 are also real. This means that the curve for $f(\sigma)$ in Fig. 1.13 will cut the σ axis at three points A, B and C in general. In the next section we shall study a few particular cases.

1.14 PARTICULAR CASES

(i) If σ_1, σ_2 and σ_3 are distinct, i.e. σ_1, σ_2 and σ_3 have different values, then the three associated principal axes $\boldsymbol{n}_1$, $\boldsymbol{n}_2$ and $\boldsymbol{n}_3$ are unique and mutually perpendicular. This follows from Eq. (1.25) of Sec. 1.12. Since σ_1, σ_2 and σ_3 are distinct, we get three distinct axes $\boldsymbol{n}_1$, $\boldsymbol{n}_2$ and $\boldsymbol{n}_3$ from Eqs (1.18), and being mutually perpendicular they are unique.

(ii) If $\sigma_1 = \sigma_2$ and σ_3 is distinct, the axis of $\boldsymbol{n}_3$ is unique and every direction perpendicular to $\boldsymbol{n}_3$ is a principal direction associated with $\sigma_1 = \sigma_2$. This is shown in Fig. 1.15.

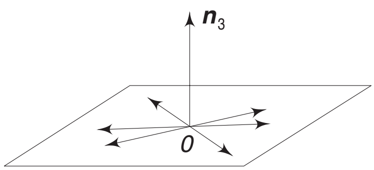

Fig. 1.15 *Case with $\sigma_1 = \sigma_2$ and σ_3 distinct*

To prove this, let us choose a frame of reference $Ox'y'z'$ such that the z' axis is along $\boldsymbol{n}_3$ and the x' and y' axes are arbitrary. From Eq. (1.29), if $\sigma_1 = \sigma_2$, then the quantity under the radical must be zero. Since this is the sum of two squared quantities, this can happen only if

$$\sigma_{x'} = \sigma_{y'} \quad \text{and} \quad \tau_{x'y'} = 0$$

But we have chosen x' and y' axes arbitrarily, and consequently the above condition must be true for any frame of reference with the z' axis along $\boldsymbol{n}_3$. Hence, the x' and y' planes are shearless planes, i.e. principal planes. Therefore, every direction perpendicular to $\boldsymbol{n}_3$ is a principal direction associated with $\sigma_1 = \sigma_2$.

(iii) If $\sigma_1 = \sigma_2 = \sigma_3$, then every direction is a principal direction. This is the hydrostatic or the isotropic state of stress and was discussed in Sec. 1.7. For proof, we can repeat the argument given in (ii). Choose a coordinate system $Ox'y'z'$ with the z' axis along $\boldsymbol{n}_3$ corresponding to σ_3. Since $\sigma_1 = \sigma_2$ every direction perpendicular to $\boldsymbol{n}_3$ is a principal direction. Next, choose the z' axis parallel to $\boldsymbol{n}_2$ corresponding to σ_2. Then every direction perpendicular to $\boldsymbol{n}_2$ is a principal direction since $\sigma_1 = \sigma_3$. Similarly, if we choose the z' axis parallel to $\boldsymbol{n}_1$ corresponding to σ_1, every direction perpendicular to $\boldsymbol{n}_1$ is also a principal direction. Consequently, every direction is a principal direction.

Another proof could be in the manner described in Sec. 1.7. Choosing $Oxyz$ coinciding with $\boldsymbol{n}_1$, $\boldsymbol{n}_2$ and $\boldsymbol{n}_3$, the stress vector on any arbitrary plane $\boldsymbol{n}$ has value σ, the direction of σ coinciding with $\boldsymbol{n}$. Hence, every plane is a principal plane. Such a state of stress is equivalent to a hydrostatic state of stress or an isotropic state of stress.

1.15 RECAPITULATION

The material discussed in the last few sections is very important and it is worthwhile to put it in the form of definitions and theorems.

Definition

For a given state of stress at point P, if the resultant stress vector $\overset{n}{\boldsymbol{T}}$ on any plane $\boldsymbol{n}$ is along $\boldsymbol{n}$ having a magnitude σ, then σ is a principal stress at P, $\boldsymbol{n}$ is the principal direction associated with σ, the axis of σ is a principal axis, and the plane is a principal plane at P.

Theorem

In every state of stress there exist at least three mutually perpendicular principal axes and at most three distinct principal stresses. The principal stresses σ_1, σ_2 and σ_3 are the roots of the cubic equation

$$\sigma^3 - l_1\sigma^2 + l_2\sigma - l_3 = 0$$

where l_1, l_2 and l_3 are the first, second and third invariants of stress. The principal directions associated with σ_1, σ_2 and σ_3 are obtained by substituting σ_i $(i = 1, 2, 3)$ in the following equations and solving for $\boldsymbol{n}_x$, $\boldsymbol{n}_y$ and $\boldsymbol{n}_z$:

$$(\sigma_x - \sigma_i)\, n_x + \tau_{xy}\, n_y + \tau_{xz}\, n_z = 0$$

$$\tau_{xy}\, n_x + (\sigma_y - \sigma_i)\, n_y + \tau_{yz}\, n_z = 0$$

$$n_x^2 + n_y^2 + n_z^2 = 1$$

If σ_1, σ_2 and σ_3 are distinct, then the axes of $\boldsymbol{n}_1$, $\boldsymbol{n}_2$ and $\boldsymbol{n}_3$ are unique and mutually perpendicular. If, say $\sigma_1 = \sigma_2 \neq \sigma_3$, then the axis of $\boldsymbol{n}_3$ is unique and every direction perpendicular to $\boldsymbol{n}_3$ is a principal direction associated with $\sigma_1 = \sigma_2$. If $\sigma_1 = \sigma_2 = \sigma_3$, then every direction is a principal direction.

Standard Method of Solution

Consider the cubic equation $y^3 + py^2 + qy + r = 0$, where p, q and r are constants.

Substitute $\quad y = x - \frac{1}{3}p$

This gives $\quad x^3 + ax + b = 0$

where $\quad a = \frac{1}{3}\left(3q - p^2\right), \qquad b = \frac{1}{27}\left(2p^3 - 9pq + 27r\right)$

Put $\quad \cos\phi = -\dfrac{b}{2\left(-\dfrac{a^3}{27}\right)^{1/2}}$

Determine ϕ, and putting $g = 2\sqrt{-a/3}$, the solutions are

$$y_1 = g\cos\frac{\phi}{3} - \frac{p}{3}$$

$$y_2 = g\cos\left(\frac{\phi}{3} + 120°\right) - \frac{p}{3}$$

$$y_3 = g\cos\left(\frac{\phi}{3} + 240°\right) - \frac{p}{3}$$

Example 1.4 *At a point P, the rectangular stress components are*

$$\sigma_x = 1, \sigma_y = -2, \quad \sigma_z = 4, \quad \tau_{xy} = 2, \quad \tau_{yz} = -3, \quad \text{and} \quad \tau_{xz} = 1$$

all in units of kPa. Find the principal stresses and check for invariance.

Solution The given stress matrix is

$$\left[\tau_{ij}\right] = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -2 & -3 \\ 1 & -3 & 4 \end{bmatrix}$$

From Eqs (1.22)–(1.24),

$$l_1 = 1 - 2 + 4 = 3$$
$$l_2 = (-2 - 4) + (-8 - 9) + (4 - 1) = -20$$
$$l_3 = 1(-8 - 9) - 2(8 + 3) + 1(-6 + 2) = -43$$

$$\therefore \quad f(\sigma) = \sigma^3 - 3\sigma^2 - 20\sigma + 43 = 0$$

For this cubic, following the standard method,

$$y = \sigma, \quad p = -3, \quad q = -20, \quad r = 43$$

$$a = \frac{1}{3}(-60 - 9) = -23$$

$$b = \frac{1}{27}(-54 - 540 + 1161) = 21$$

$$\cos \phi = -\frac{\left(\frac{21}{2}\right)}{\left(\frac{12167}{27}\right)^{1/2}}$$

$$\therefore \quad \phi = -119° \, 40$$

The solutions are

$$\sigma_1 = y_1 = 4.25 + 1 = 5.25 \text{ kPa}$$
$$\sigma_2 = y_2 = -5.2 + 1 = -4.2 \text{ kPa}$$
$$\sigma_3 = y_3 = 0.95 + 1 = 1.95 \text{ kPa}$$

Renaming such that $\sigma_1 \geq \sigma_2 \geq \sigma_3$ we have,

$$\sigma_1 = 5.25 \text{ kPa}, \quad \sigma_2 = 1.95 \text{ kPa}, \quad \sigma_3 = -4.2 \text{ kPa}$$

The stress invariants are

$$l_1 = 5.25 + 1.95 - 4.2 = 3.0$$
$$l_2 = (5.25 \times 1.95) - (1.95 \times 4.2) - (4.2 \times 5.25) = -20$$
$$l_3 = -(5.25 \times 1.95 \times 4.2) = -43$$

These agree with their earlier values.

Example 1.5 *With respect to the frame of reference* Oxyz, *the following state of stress exists. Determine the principal stresses and their associated directions. Also, check on the invariances of* l_1, l_2, l_3.

$$[\tau_{ij}] = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Solution For this state

$$I_1 = 1 + 1 + 1 = 3$$
$$I_2 = (1 - 4) + (1 - 1) + (1 - 1) = -3$$
$$I_3 = 1(1 - 1) - 2(2 - 1) + 1(2 - 1) = -1$$
$$f(\sigma) = \sigma^3 - I_1 \sigma^2 + I_2 \sigma - I_3 = 0$$

i.e., $\sigma^3 - 3\sigma^2 - 3\sigma + 1 = 0$

or $(\sigma^3 + 1) - 3\sigma(\sigma + 1) = 0$

i.e., $(\sigma + 1)(\sigma^2 - \sigma + 1) - 3\sigma(\sigma + 1) = 0$

or $(\sigma + 1)(\sigma^2 - 4\sigma + 1) = 0$

Hence, one solution is $\sigma = -1$. The other two solutions are obtained from the solution of the quadratic equation, which are $\sigma = 2 \pm \sqrt{3}$.

$$\therefore \quad \sigma_1 = -1, \quad \sigma_2 = 2 + \sqrt{3}, \quad \sigma_3 = 2 - \sqrt{3}$$

Check on the invariance:
With the set of axes chosen along the principal axes, the stress matrix will have the form

$$[\tau_{ij}] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2+\sqrt{3} & 0 \\ 0 & 0 & 2-\sqrt{3} \end{bmatrix}$$

Hence, $I_1 = -1 + 2 + \sqrt{3} + 2 - \sqrt{3} = 3$

$$I_2 = (-2 - \sqrt{3}) + (4 - 3) + (-2 + \sqrt{3}) = -3$$
$$I_3 = -1\,(4 - 3) = -1$$

Directions of principal axes:

(i) For $\sigma_1 = -1$, from Eqs (1.18) and (1.21)

$$(1 + 1)n_x + 2n_y + n_z = 0$$
$$2n_x + (1 + 1)n_y + n_z = 0$$
$$n_x + n_y + (1 + 1)n_z = 0$$

together with

$$n_x^2 + n_y^2 + n_z^2 = 1$$

From the second and third equations above, $n_z = 0$. Using this in the third and fourth equations and solving, $n_x = \pm\left(1/\sqrt{2}\right)$, $n_y = \pm\left(1/\sqrt{2}\right)$.

Hence, $\sigma_1 = -1$ is in the direction $\left(+1/\sqrt{2}, -1/\sqrt{2}, 0\right)$.

It should be noted that the plus and minus signs associated with n_x, n_y and n_z represent the same line.

(ii) For $\sigma_2 = 2 + \sqrt{3}$

$$(-1-\sqrt{3})n_x + 2n_y + n_z = 0$$
$$2n_x + (-1-\sqrt{3})n_y + n_z = 0$$
$$n_x + n_y(-1-\sqrt{3})n_z = 0$$

together with

$$n_x^2 + n_y^2 + n_z^2 = 1$$

Solving, we get

$$n_x = n_y = \left(1+\frac{1}{\sqrt{3}}\right)^{1/2} \qquad n_z = \frac{1}{\left(3+\sqrt{3}\right)^{1/2}}$$

(iii) For $\sigma_3 = 2 - \sqrt{3}$
We can solve for n_x, n_y and n_z in a manner similar to the preceeding one or get the solution from the condition that $\boldsymbol{n}_1$, $\boldsymbol{n}_2$ and $\boldsymbol{n}_3$ form a right-angled triad, i.e. $\boldsymbol{n}_3 = \boldsymbol{n}_1 \times \boldsymbol{n}_2$.
The solution is

$$n_x = n_y = -\frac{1}{2}\left(1-\frac{1}{\sqrt{3}}\right)^{1/2}, \quad n_z = \frac{1}{\sqrt{2}}\left(1+\frac{1}{\sqrt{3}}\right)^{1/2}$$

Example 1.6 *For the given state of stress, determine the principal stresses and their directions.*

$$[\tau_{ij}] = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Solution $\qquad I_1 = 0,\ I_2 = -3,\ I_3 = 2$

$$f(\sigma) = -\sigma^3 + 3\sigma + 2 = 0$$
$$= (-\sigma^3 - 1) + (3\sigma + 3)$$
$$= -(\sigma + 1)(\sigma^2 - \sigma + 1) + 3(\sigma + 1)$$
$$= (\sigma + 1)(\sigma - 2)(\sigma + 1) = 0$$

$\therefore \qquad \sigma_1 = \sigma_2 = -1 \quad \text{and} \quad \sigma_3 = 2$

Since two of the three principal stresses are equal, and σ_3 is different, the axis of σ_3 is unique and every direction perpendicular to σ_3 is a principal direction associated with $\sigma_1 = \sigma_2$. For $\sigma_3 = 2$

$$-2n_x + n_y + n_z = 0$$
$$n_x - 2n_y + n_z = 0$$

$$n_x + n_y - 2n_z = 0$$
$$n_x^2 + n_y^2 + n_z^2 = 1$$

These give $n_x = n_y = n_z = \dfrac{1}{\sqrt{3}}$

Example 1.7 *The state of stress at a point is such that*

$$\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = \tau_{yz} = \tau_{zx} = \rho$$

Determine the principal stresses and their directions

Solution For the given state,

$$l_1 = 3\rho, \qquad l_2 = 0, \qquad l_3 = 0$$

Therefore the cubic is $\sigma^3 - 3\rho\sigma^2 = 0$; the solutions are $\sigma_1 = 3\rho$, $\sigma_2 = \sigma_3 = 0$. For $\sigma_1 = 3\rho$

$$(\rho - 3\rho)n_x + \rho n_y + \rho n_z = 0$$
$$\rho n_x + (\rho - 3p)\, n_y + \rho n_z = 0$$
$$\rho n_x + \rho n_y + (\rho - 3\rho)\, n_z = 0$$

or

$$-2n_x + n_y + n_z = 0$$
$$n_x - 2n_y + n_z = 0$$
$$n_x + n_y - 2n_z = 0$$

The above equations give

$$n_x = n_y = n_z$$

With $n_x^2 + n_y^2 + n_z^2 = 1$, one gets $n_x = n_y = n_z = 1/\sqrt{3}$.

Thus, on a plane that is equally inclined to xyz axes, there is a tensile stress of magnitue 3ρ. This is the case of a uniaxial tension, the axis of loading making equal angles with the given xyz axes. If one denotes this loading axis by z', the other two axes, x' and y', can be chosen arbitrarily, and the planes normal to these, i.e. x' plane and y' plane, are stress free.

1.16 THE STATE OF STRESS REFERRED TO PRINCIPAL AXES

In expressing the state of stress at a point by the six rectangular stress components, we can choose the principal axes as the coordinate axes and refer the rectangular stress components accordingly. We then have for the stress matrix

$$[\tau_{ij}] = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \tag{1.30}$$

On any plane with normal $\boldsymbol{n}$, the components of the stress vector are, from Eq. (1.9),

$$\overset{n}{\boldsymbol{T}}_x = \sigma_1 n_x, \qquad \overset{n}{\boldsymbol{T}}_y = \sigma_2 n_y, \qquad \overset{n}{\boldsymbol{T}}_z = \sigma_3 n_z \tag{1.31}$$

The resultant stress has a magnitutde

$$\left|\overset{n}{\boldsymbol{T}}\right|^2 = \sigma_1^2 n_x^2 + \sigma_2^2 n_y^2 + \sigma_3^2 n_z^2 \tag{1.32}$$

If σ is the normal and τ the shearing stress on this plane, then

$$\sigma = \sigma_1 n_x^2 + \sigma_2 n_y^2 + \sigma_3 n_z^2 \tag{1.33}$$

and $$\tau^2 = \left|\overset{n}{\boldsymbol{T}}\right|^2 - \sigma^2 \tag{1.34}$$

$$= n_x^2 n_y^2 (\sigma_1 - \sigma_2)^2 + n_y^2 n_z^2 (\sigma_2 - \sigma_3)^2 + n_z^2 n_x^2 (\sigma_3 - \sigma_1)^2$$

The stress invariants assume the form

$$\begin{aligned} l_1 &= \sigma_1 + \sigma_2 + \sigma_3 \\ l_2 &= \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 \\ l_3 &= \sigma_1\sigma_2\sigma_3 \end{aligned} \tag{1.35}$$

1.17 MOHR'S CIRCLES FOR THE THREE-DIMENSIONAL STATE OF STRESS

We shall now describe a geometrical construction that brings out some important results. At a given point P, let the frame of reference $Pxyz$ be chosen along the principal stress axes. Consider a plane with normal $\boldsymbol{n}$ at point P. Let σ be the normal stress and τ the shearing stress on this plane. Take another set of axes σ and τ. In this plane we can mark a point Q with co-ordinates (σ, τ) representing the values of the normal and shearing stress on the plane $\boldsymbol{n}$. For different planes passing through point P, we get different values of σ and τ. Corresponding to each plane $\boldsymbol{n}$, a point Q can be located with coordinates (σ, τ). The plane with the σ axis and the τ axis is called the stress plane π. (No numerical value is associated with this symbol). The problem now is to determine the bounds for Q (σ, τ) for all possible directions $\boldsymbol{n}$.

Arrange the principal stresses such that algebraically

$$\sigma_1 \geq \sigma_2 \geq \sigma_3$$

Mark off σ_1, σ_2 and σ_3 along the σ axis and construct three circles with diameters $(\sigma_1 - \sigma_2)$, $(\sigma_2 - \sigma_3)$ and $(\sigma_1 - \sigma_3)$ as shown in Fig. 1.16.

It will be shown in Sec. 1.18 that the point $Q(\sigma, \tau)$ for all possible $\boldsymbol{n}$ will lie within the shaded area. This region is called Mohr's stress plane π and the three circles are known as Mohr's circles. From Fig. 1.16, the following points can be observed:

(i) Points A, B and C represent the three principal stresses and the associated shear stresses are zero.

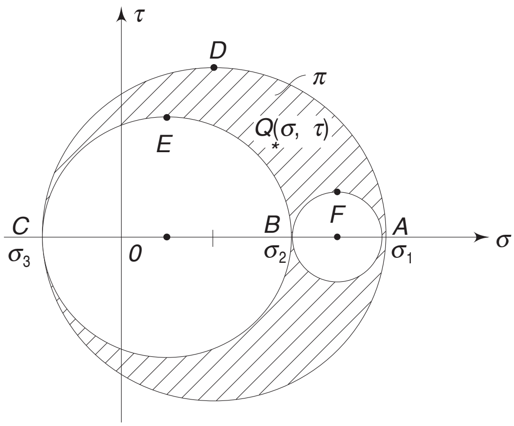

Fig. 1.16 *Mohr's stress plane*

(ii) The maximum shear stress is equal to $\frac{1}{2}(\sigma_1 - \sigma_3)$ and the associated normal stress is $\frac{1}{2}(\sigma_1 + \sigma_3)$. This is indicated by point *D* on the outer circle.

(iii) Just as there are three extremum values σ_1, σ_2 and σ_3 for the normal stresses, there are three extremum values for the shear stresses, these being $\frac{\sigma_1 - \sigma_3}{2}$, $\frac{\sigma_2 - \sigma_3}{2}$ and $\frac{\sigma_1 - \sigma_2}{2}$. The planes on which these shear stresses act are called the principal shear planes. While the planes on which the principal normal stresses act are free of shear stresses, the principal shear planes are not free from normal stresses. The normal stresses associated with the principal shears are respectively $\frac{\sigma_1 + \sigma_3}{2}$, $\frac{\sigma_2 + \sigma_3}{2}$ and $\frac{\sigma_1 + \sigma_2}{2}$. These are indicated by points *D*, *E* and *F* in Fig. 1.16. It will be shown in Sec. 1.19 that the principal shear planes are at 45° to the principal normal planes. The principal shears are denoted by τ_1, τ_2 and τ_3 where

$$2\tau_3 = (\sigma_1 - \sigma_2), \qquad 2\tau_2 = (\sigma_1 - \sigma_3), \qquad 2\tau_1 = (\sigma_2 - \sigma_3) \tag{1.36}$$

(iv) When $\sigma_1 = \sigma_2 \neq \sigma_3$ or $\sigma_1 \neq \sigma_2 = \sigma_3$, the three circles reduce to only one circle and the shear stress on any plane will not exceed $\frac{1}{2}(\sigma_1 - \sigma_3)$ or $\frac{1}{2}(\sigma_1 - \sigma_2)$ according as $\sigma_1 = \sigma_2$ or $\sigma_2 = \sigma_3$.

(v) When $\sigma_1 = \sigma_2 = \sigma_3$, the three circles collapse to a single point on the σ axis and every plane is a shearless plane.

1.18 MOHR'S STRESS PLANE

It was stated in the previous section that when points with coordinates (σ, τ) for all possible planes passing through a point are marked on the $\sigma - \tau$ plane, as in Fig. 1.16, the points are bounded by the three Mohr's circles. In this, section we shall prove this.

Choose the coordinate frame of reference *Pxyz* such that the axes are along the principal axes. On any plane with normal $\mathbf{n}$, the resultant stress vector $\overset{n}{\mathbf{T}}$ and the normal stress σ are such that from Eqs (1.32) and (1.33)

$$\left|\overset{n}{\mathbf{T}}\right|^2 = \sigma^2 + \tau^2 = \sigma_1^2 n_x^2 + \sigma_2^2 n_y^2 + \sigma_3^2 n_z^2 \tag{1.37}$$

$$\sigma = \sigma_1 n_x^2 + \sigma_2 n_y^2 + \sigma_3 n_z^2 \tag{1.38}$$

and also

$$1 = n_x^2 + n_y^2 + n_z^2 \tag{1.39}$$

The above three equations can be used to solve for n_x^2, n_y^2 and n_z^2 yielding

$$n_x^2 = \frac{(\sigma - \sigma_2)(\sigma - \sigma_3) + \tau^2}{(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3)} \tag{1.40}$$

$$n_y^2 = \frac{(\sigma - \sigma_3)(\sigma - \sigma_1) + \tau^2}{(\sigma_2 - \sigma_3)(\sigma_2 - \sigma_1)} \tag{1.41}$$

$$n_z^2 = \frac{(\sigma - \sigma_1)(\sigma - \sigma_2) + \tau^2}{(\sigma_3 - \sigma_1)(\sigma_3 - \sigma_2)} \tag{1.42}$$

Since n_x^2, n_y^2 and n_z^2 are all positive, the right-hand side expressions in the above equations must all be positive. Recall that we have arranged the principal stresses such that $\sigma_1 \geq \sigma_2 \geq \sigma_3$. There are three cases one can consider.

Case (i) $\sigma_1 > \sigma_2 > \sigma_3$

Case (ii) $\sigma_1 = \sigma_2 > \sigma_3$

Case (iii) $\sigma_1 = \sigma_2 = \sigma_3$

We shall consider these cases individually.

Case (i) $\sigma_1 > \sigma_2 > \sigma_3$

For this case, the denominator in Eq. (1.40) is positive and hence, the numerator must also be positive. In Eq. (1.41), the denominator being negative, the numerator must also be negative. Similarly, the numerator in Eq. (1.42) must be positive. Therefore.

$$(\sigma - \sigma_2)(\sigma - \sigma_3) + \tau^2 \geq 0$$
$$(\sigma - \sigma_3)(\sigma - \sigma_1) + \tau^2 \leq 0$$
$$(\sigma - \sigma_1)(\sigma - \sigma_2) + \tau^2 \geq 0$$

The above three inequalities can be rewritten as

$$\tau^2 + \left(\sigma - \frac{\sigma_2 + \sigma_3}{2}\right)^2 \geq \left(\frac{\sigma_2 - \sigma_3}{2}\right)^2$$

$$\tau^2 + \left(\sigma - \frac{\sigma_3 + \sigma_1}{2}\right)^2 \le \left(\frac{\sigma_3 - \sigma_1}{2}\right)^2$$

$$\tau^2 + \left(\sigma - \frac{\sigma_1 + \sigma_2}{2}\right)^2 \ge \left(\frac{\sigma_1 - \sigma_2}{2}\right)^2$$

According to the first of the above equations, the point (σ, τ) must lie on or outside a circle of radius $\frac{1}{2}(\sigma_2 - \sigma_3)$ with its centre at $\frac{1}{2}(\sigma_2 + \sigma_3)$ along the σ axis (Fig. 1.16). This is the circle with *BC* as diameter. The second equation indicates that the point (σ, τ) must lie inside or on the circle *ADC* with radius $\frac{1}{2}(\sigma_1 - \sigma_3)$ and centre at $\frac{1}{2}(\sigma_1 + \sigma_3)$ on the σ axis. Similarly, the last equation indicates that the point (σ, τ) must lie on or outside the circle *AFB* with radius equal to $\frac{1}{2}(\sigma_1 - \sigma_2)$ and centre at $\frac{1}{2}(\sigma_1 + \sigma_2)$.

Hence, for this case, the point $Q(\sigma, \tau)$ should lie inside the shaded area of Fig. 1.16.

Case (ii) $\sigma_1 = \sigma_2 > \sigma_3$

Following arguments similar to the ones given above, one has for this case from Eqs (1.40)–(1.42)

$$\tau^2 + \left(\sigma - \frac{\sigma_2 + \sigma_3}{2}\right)^2 = \left(\frac{\sigma_2 - \sigma_3}{2}\right)^2$$

$$\tau^2 + \left(\sigma - \frac{\sigma_3 + \sigma_1}{2}\right)^2 = \left(\frac{\sigma_3 - \sigma_1}{2}\right)^2$$

$$\tau^2 + \left(\sigma - \frac{\sigma_1 + \sigma_2}{2}\right)^2 \ge \left(\frac{\sigma_1 - \sigma_2}{2}\right)^2$$

From the first two of these equations, since $\sigma_1 = \sigma_2$, point (σ, τ) must lie on the circle with radius $\frac{1}{2}(\sigma_1 - \sigma_3)$ with its centre at $\frac{1}{2}(\sigma_1 + \sigma_3)$. The last equation indicates that the point must lie outside a circle of zero radius (since $\sigma_1 = \sigma_2$). Hence, in this case, the Mohr's circles will reduce to a circle *BC* and a point circle *B*. The point *Q* lies on the circle *BEC*.

Case (iii) $\sigma_1 = \sigma_2 = \sigma_3$

This is a trivial case since this is the isotropic or the hydrostatic state of stress. Mohr's circles collapse to a single point on the σ axis.

See Appendix 1 for the graphical determination of the normal and shear stresses on an arbitrary plane, using Mohr's circles.

1.19 PLANES OF MAXIMUM SHEAR

From Sec. 1.17 and also from Fig. 1.16 for the case $\sigma_1 > \sigma_2 > \sigma_3$, the maximum shear stress is $\frac{1}{2}(\sigma_1 - \sigma_3) = \tau_2$ and the associated normal stress is $\frac{1}{2}(\sigma_1 + \sigma_3)$.

Substituting these values in Eqs.(1.37)–(1.39) in Sec. 1.18, one gets $n_x = \pm\sqrt{1/2}$, $n_y = 0$ and $n_z = \pm 1\sqrt{2}$. This means that the planes (there are two of them) on which the shear stress takes on an extremum value, make angles of 45° and 135° with the σ_1 and σ_2 planes as shown in Fig. 1.17.

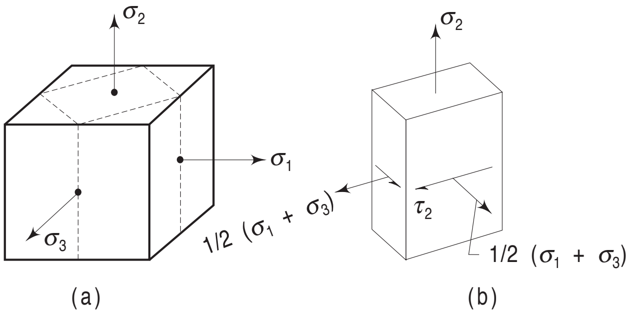

Fig. 1.17 *(a) Principal planes (b) Planes of maximum shear*

If $\sigma_1 = \sigma_2 > \sigma_3$, then the three Mohr's circles reduce to one circle *BC* (Fig.1.16) and the maximum shear stress will be $\frac{1}{2}(\sigma_2 - \sigma_3) = \tau_{1,}$ with the associated normal stress $\frac{1}{2}(\sigma_2 + \sigma_3)$. Substituting these values in Eqs (1.37)–(1.39), we get $n_x = 0/0$, $n_y = 0/0$ and $n_z = \pm 1\sqrt{2}$ i.e. n_x and n_y are indeterminate. This means that the planes on which τ_1 is acting makes angles of 45° and 135° with the σ_3 axis but remains indeterminate with respect to σ_1 and σ_2 axes. This is so because, since $\sigma_1 = \sigma_2 \neq \sigma_3$, the axis of σ_3 is unique, whereas, every direction perpendicular to σ_3 is a principal direction associated with $\sigma_1 = \sigma_2$ (Sec. 1.14). The principal shear plane will, therefore, make a fixed angle with σ_3 axis (45° or 135°) but will have different values depending upon the selection of σ_1 and σ_3 axes.

1.20 OCTAHEDRAL STRESSES

Let the frame of reference be again chosen along σ_1, σ_2 and σ_3 axes. A plane that is equally inclined to these three axes is called an octahedral plane. Such a plane will have $n_x = n_y = n_z$. Since $n_x^2 + n_y^2 + n_z^2 = 1$, an octahedral plane will be defined by $n_x = n_y = n_z = \pm 1/\sqrt{3}$. There are eight such planes, as shown in Fig.1.18.

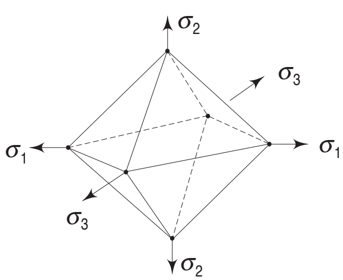

Fig.1.18 *Octahedral planes*

The normal and shearing stresses on these planes are called the octahedral normal stress and octahedral shearing stress respectively. Substituting $n_x = n_y = n_z = \pm 1/\sqrt{3}$ in Eqs (1.33) and (1.34),

$$\sigma_{oct} = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{1}{3} I_1 \tag{1.43}$$

and
$$\tau_{oct}^2 = \frac{1}{9}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] \tag{1.44a}$$

or
$$9\tau_{oct}^2 = 2(\sigma_1 + \sigma_2 + \sigma_3)^2 - 6(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) \tag{1.44b}$$

or
$$\tau_{oct} = \frac{\sqrt{2}}{3}(I_1^2 - 3I_2)^{1/2} \tag{1.44c}$$

It is important to remember that the octahedral planes are defined with respect to the principal axes and not with reference to an arbitrary frame of reference. Since σ_{oct} and τ_{oct} have been expressed in terms of the stress invariants, one can express these in terms of σ_x, σ_y, σ_z, τ_{xy}, τ_{yz} and τ_{zx} also. Using Eqs (1.22) and (1.23),

$$\sigma_{oct} = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z) \tag{1.45}$$

$$9\tau_{oct}^2 = (\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6\left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2\right) \tag{1.46}$$

The octahedral normal stress being equal to 1/3 I_1, it may be interpreted as the mean normal stress at a given point in a body. If in a state of stress, the first invariant $(\sigma_1 + \sigma_2 + \sigma_3)$ is zero, then the normal stresses on the octahedral planes will be zero and only the shear stresses will act. This is important from the point of view of the strength and failure of some materials (see Chapter 4).

Example 1.8 *The state of stress at a point is characterised by the components*

$$\sigma_x = 100 \text{ MPa}, \sigma_y = -40 \text{ MPa}, \sigma_z = 80 \text{ MPa},$$
$$\tau_{xy} = \tau_{yz} = \tau_{zx} = 0$$

Determine the extremum values of the shear stresses, their associated normal stresses, the octahedral shear stress and its associated normal stress.

Solution The given stress components are the principal stresses, since the shears are zero. Arranging the terms such that $\sigma_1 \geq \sigma_2 \geq \sigma_3$,

$$\sigma_1 = 100 \text{ MPa}, \sigma_2 = 80 \text{ MPa}, \sigma_3 = -40 \text{ MPa}$$

Hence from Eq. (1.36),

$$\tau_1 = \frac{\sigma_2 - \sigma_3}{2} = \frac{80 + 40}{2} = 60 \text{ MPa}$$

$$\tau_2 = \frac{\sigma_3 - \sigma_1}{2} = \frac{-40 - 100}{2} = -70 \text{ MPa}$$

$$\tau_3 = \frac{\sigma_1 - \sigma_2}{2} = \frac{100 - 80}{2} = 10 \text{ MPa}$$

The associated normal stresses are

$$\sigma_1^* = \frac{\sigma_2 + \sigma_3}{2} = \frac{80 - 40}{2} = 20 \text{ MPa}$$

$$\sigma_2^* = \frac{\sigma_3 + \sigma_1}{2} = \frac{-40 + 100}{2} = 30 \text{ MPa}$$

$$\sigma_3^* = \frac{\sigma_1 + \sigma_2}{2} = \frac{100 + 80}{2} = 90 \text{ MPa}$$

$$\tau_{oct} = \frac{1}{3}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2} = 61.8 \text{ MPa}$$

$$\sigma_{oct} = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{140}{3} = 46.7 \text{ MPa}$$

1.21 THE STATE OF PURE SHEAR

The state of stress at a point can be characterised by the six rectangular stress components referred to a coordinate frame of reference. The magnitudes of these components depend on the choice of the coordinate system. If, for at least one particular choice of the frame of reference, we find that $\sigma_x = \sigma_y = \sigma_z = 0$, then a state of pure shear is said to exist at point P. For such a state, with that particular choice of coordinate system, the stress matrix will be

$$[\tau_{ij}] = \begin{bmatrix} 0 & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & 0 & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & 0 \end{bmatrix}$$

For this coordinate system, $l_1 = \sigma_x + \sigma_y + \sigma_z = 0$. Since l_1 is an invariant, this must be true for any choice of coordinate system selected at P. Hence, the necessary condition for a state of pure shear to exist is that $l_1 = 0$, It can be shown (Appendix 2) that this is also a sufficient condition.

It was remarked in the previous section that when $l_1 = 0$, an octahedral plane is subjected to pure shear with no normal stress. Hence, for a pure shear stress state, the octahedral plane (remember that this plane is defined with respect to the principal axes and not with respect to an arbitrary set of axes) is free from normal stress.

1.22 DECOMPOSITION INTO HYDROSTATIC AND PURE SHEAR STATES

It will be shown in the present section that an arbitrary state of stress can be resolved into a hydrostatic state and a state of pure shear. Let the given state referred to a coordinate system be

$$[\tau_{ij}] = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}$$

Let $$p = 1/3(\sigma_x + \sigma_y + \sigma_z) = 1/3 l_1 \tag{1.47}$$

The given state can be resolved into two different states, as shown:

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} = \begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix} + \begin{bmatrix} \sigma_x - p & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y - p & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z - p \end{bmatrix} \tag{1.48}$$

The first state on the right-hand side of the above equation is a hydrostatic state. [Refer Sec. 1.14(iii).]

The second state is a state of pure shear since the first invariant for this state is

$$\begin{aligned} l'_1 &= (\sigma_x - p) + (\sigma_y - p) + (\sigma_x - p) \\ &= \sigma_x + \sigma_y + \sigma_z - 3p \\ &= 0 \text{ from Eq. (1.47)} \end{aligned}$$

If the given state is referred to the principal axes, the decomposition into a hydrostatic state and a pure shear state can once again be done as above, i.e.

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} = \begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix} + \begin{bmatrix} \sigma_1 - p & 0 & 0 \\ 0 & \sigma_2 - p & 0 \\ 0 & 0 & \sigma_3 - p \end{bmatrix} \tag{1.49}$$

where, as before, $p = 1/3(\sigma_1 + \sigma_2 + \sigma_3) = 1/3 l_1$.

The pure shear state of stress is also known as the deviatoric state of stress or simply as stress deviator.

Example 1.9 *The state of stress characterised by τ_{ij} is given below. Resolve the given state into a hydrostatic state and a pure shear state. Determine the normal and shearing stresses on an octahedral plane. Compare these with the σ_{oct} and τ_{oct} calculated for the hydrostatic and the pure shear states. Are the octahedral planes for the given state, the hydrostatic state and the pure shear state the same or are they different? Explain why.*

$$[\tau_{ij}] = \begin{bmatrix} 10 & 4 & 6 \\ 4 & 2 & 8 \\ 6 & 8 & 6 \end{bmatrix}$$

Solution $\quad l_1 = 10 + 2 + 6 = 18, \quad \dfrac{1}{3} l_1 = 6$

Resolving into hydrostatic and pure shear state, Eq. (1.47),

$$[\tau_{ij}] = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} + \begin{bmatrix} 4 & 4 & 6 \\ 4 & -4 & 8 \\ 6 & 8 & 0 \end{bmatrix}$$

For the given state, the octahedral normal and shear stresses are:

$$\sigma_{oct} = \frac{1}{3} I_1 = 6$$

From Eq. (1.44)

$$\tau_{oct} = \frac{\sqrt{2}}{3}\left(I_1^2 - 3I_2^2\right)^{1/2}$$

$$= \frac{\sqrt{2}}{3}\left[18^2 - 3\left(20 - 16 + 12 - 64 + 60 - 36\right)\right]^{1/2}$$

$$= \frac{\sqrt{2}}{3}(396)^{1/2} = 2\sqrt{22}$$

For the hydrostatic state, $\sigma_{oct} = 6$, since every plane is a principal plane with $\sigma = 6$ and consequently, $\tau_{oct} = 0$.

For the pure shear state, $\sigma_{oct} = 0$ since the first invariant of stress for the pure shear state is zero. The value of the second invariant of stress for the pure shear state is

$$I'_2 = (-16 - 16 + 0 - 64 + 0 - 36\ ) = -132$$

Hence, the value of τ_{oct} for the pure shear state is

$$\tau_{oct} = \frac{\sqrt{2}}{3}(396)^{1/2} = 2\sqrt{22}$$

Hence, the value of σ_{oct} for the given state is equal to the value of σ_{oct} for the hydrostatic state, and τ_{oct} for the given state is equal to τ_{oct} for the pure shear state.

The octahedral planes for the given state (which are identified after determining the principal stress directions), the hydrostatic state and the pure shear state are all identical. For the hydrostatic state, every direction is a principal direction, and hence, the principal stress directions for the given state and the pure shear state are identical. Therefore, the octahedral planes corresponding to the given state and the pure shear state are identical.

Example 1.10 *A cylindrical boiler, 180 cm in diameter, is made of plates 1.8 cm thick, and is subjected to an internal pressure 1400 kPa. Determine the maximum shearing stress in the plate at point P and the plane on which it acts.*

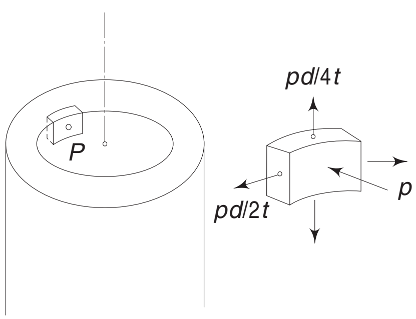

Fig. 1.19 *Example 1.10*

Solution From elementary strength of materials, the axial stress in the plate is $\frac{pd}{4t}$ where p is the internal pressure, d the diameter and t the thickness. The circumferential or the hoop stress is $\frac{pd}{2t}$.

The state of stress acting on an element is as shown in Fig. 1.19.

The principal stresses when arranged such that $\sigma_1 \geq \sigma_2 \geq \sigma_3$ are

$$\sigma_1 = \frac{pd}{2t}; \qquad \sigma_2 = \frac{pd}{4t}; \qquad \sigma_3 = -p$$

The maximum shear stress is therefore,

$$\tau_{\max} = \frac{1}{2}(\sigma_1 - \sigma_3) = \frac{1}{2}p\left(\frac{d}{2t}+1\right)$$

Substituting the values

$$\tau_{\max} = \frac{1400}{2}\left(\frac{1.8\times 100}{2\times 1.8}+1\right) = 35,700 \text{ kPa}$$

1.23 CAUCHY'S STRESS QUADRIC

We shall now describe a geometrical description of the state of stress at a point P. Choose a frame of reference whose axes are along the principal axes. Let σ_1, σ_2 and σ_3 be the principal stresses. Consider a plane with normal $\boldsymbol{n}$. The normal stress on this plane is from Eq. (1.33),

$$\sigma = \sigma_1 n_x^2 + \sigma_2 n_y^2 + \sigma_3 n_z^2$$

Along the normal $\boldsymbol{n}$ to the plane, choose a point Q such that

$$PQ = R = 1/\sqrt{\sigma} \tag{1.50}$$

As different planes $\boldsymbol{n}$ are chosen at P, we get different values for the normal stress σ and correspondingly different PQs. If such Qs are marked for every plane passing through P, then we get a surface S. This surface determines the normal component of stress on every plane passing through P. This surface is known as the stress surface of Cauchy. This has a very interesting property. Let Q be a point on the surface, Fig. 1.20(a). By the previous definition, the length $PQ = R$ is such that the normal stress on the plane whose normal is along PQ is given by

$$\sigma = \frac{1}{R^2} \tag{1.51}$$

If $\boldsymbol{m}$ is a normal to the tangent plane to the surface S at point Q, then this normal $\boldsymbol{m}$ is parallel to the resultant stress vector $\overset{n}{\boldsymbol{T}}$ at P.

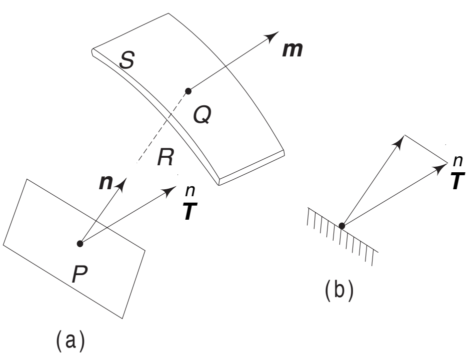

Fig. 1.20 *(a) Cauchy's stress quadric (b) Resultant stress vector and normal stress component*

Since the direction of the resultant vector $\overset{n}{\boldsymbol{T}}$ is known, and its component σ along the normal is known, the resultant stress vector $\overset{n}{\boldsymbol{T}}$ can be easily determined, as shown in Fig. 1.20(b).

We shall now show that the normal $\boldsymbol{m}$ to the surface S is parallel to $\overset{n}{\boldsymbol{T}}$, the resultant stress vector. Let $Pxyz$ be the principal axes at P (Fig. 1.21). $\boldsymbol{n}$ is the normal to a particular plane at P. The normal stress on this plane, as before, is

$$\sigma = \sigma_1 n_x^2 + \sigma_2 n_y^2 + \sigma_3 n_z^2$$

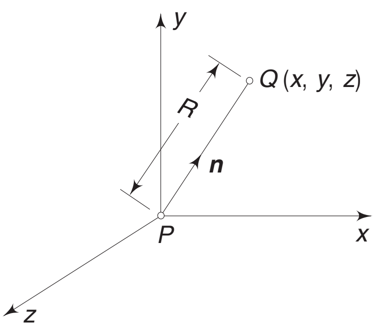

Fig. 1.21 *Principal axes at P and* ***n*** *to a plane*

If the coordinates of the point Q are (x, y, z) and the length $PQ = R$, then

$$n_x = \frac{x}{R}, \qquad n_y = \frac{y}{R}, \qquad n_z = \frac{z}{R} \tag{1.52}$$

Substituting these in the above equation for σ

$$\sigma R^2 = \sigma_1 x^2 + \sigma_2 y^2 + \sigma_3 z^2$$

From Eq. (1.51), we have $\sigma R^2 = \pm 1$. The plus sign is used when σ is tensile and the minus sign is used when σ is compressive. Hence, the surface S has the equations (a surface of second degree)

when σ is tensile

$$\sigma_1 x^2 + \sigma_2 y^2 + \sigma_3 z^2 = +1 \tag{1.53a}$$

when σ is compressive

$$\sigma_1 x^2 + \sigma_2 y^2 + \sigma_3 z^2 = -1 \tag{1.53b}$$

We know from calculus that for a surface with equation $F(x, y, z) = 0$, the normal to the tangent plane at a point Q on the surface has direction cosines proportional to $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}$ and $\frac{\partial F}{\partial z}$. From Fig. (1.20), $\boldsymbol{m}$ is the normal perpendicular to the tangent plane to S at Q. Hence, if m_x, m_y, and m_z are the direction cosines of $\boldsymbol{m}$, then

$$m_x = \alpha \frac{\partial F}{\partial x}, \qquad m_y = \alpha \frac{\partial F}{\partial y}, \qquad m_z = \alpha \frac{\partial F}{\partial z}$$

From Eq. (1.53a) or Eq. (1.53b)

$$m_x = 2\alpha\sigma_1 x, \qquad m_y = 2\alpha\sigma_2 y, \qquad m_z = 2\alpha\sigma_3 z \tag{1.54}$$

where α is a constant of proportionality.

$\overset{n}{\boldsymbol{T}}$ is the resultant stress vector on plane $\boldsymbol{n}$ and its components $\overset{n}{\boldsymbol{T}}_x, \overset{n}{\boldsymbol{T}}_y$, and $\overset{n}{\boldsymbol{T}}_z$ according to Eq. (1.31), are

$$\overset{n}{\boldsymbol{T}}_x = \sigma_1 \boldsymbol{n}_x, \qquad \overset{n}{\boldsymbol{T}}_y = \sigma_1 \boldsymbol{n}_y, \qquad \overset{n}{\boldsymbol{T}}_z = \sigma_3 \boldsymbol{n}_z$$

Substituting for n_x, n_y and n_z from Eq. (1.52)

$$\overset{n}{\boldsymbol{T}}_x = \frac{1}{R}\sigma_1 x, \qquad \overset{n}{\boldsymbol{T}}_y = \frac{1}{R}\sigma_2 y, \qquad \overset{n}{\boldsymbol{T}}_z = \frac{1}{R}\sigma_3 z$$

or $$\sigma_1 x = R\overset{n}{\boldsymbol{T}}_x, \qquad \sigma_2 y = R\overset{n}{\boldsymbol{T}}_y, \qquad \sigma_3 z = R\overset{n}{\boldsymbol{T}}_z$$

Substituting these in Eq. (1.54)

$$m_x = 2\alpha\, R\overset{n}{\boldsymbol{T}}_x, \qquad m_y = 2\alpha\, R\overset{n}{\boldsymbol{T}}_y, \qquad m_z = 2\alpha\, R\overset{n}{\boldsymbol{T}}_z$$

i.e. m_x, m_y and m_z are proportional to $\overset{n}{\boldsymbol{T}}_x$, $\overset{n}{\boldsymbol{T}}_y$ and $\overset{n}{\boldsymbol{T}}_z$.

Hence, $\boldsymbol{m}$ and $\overset{n}{\boldsymbol{T}}$ are parallel.

The stress surface of Cauchy, therefore, has the following properties:

(i) If Q is a point on the stress surface, then $PQ = 1/\sqrt{\sigma}$ where σ is the normal stress on a plane whose normal is PQ.

(ii) The normal to the surface at Q is parallel to the resultant stress vector $\overset{n}{\boldsymbol{T}}$ on the plane with normal PQ.

Therefore, the stress surface of Cauchy completely defines the state of stress at P. It would be of interest to know the shape of the stress surface for different states of stress. This aspect will be discussed in Appendix 3.

1.24 LAME'S ELLIPSOID

Let $Pxyz$ be a coordinate frame of reference at point P, parallel to the principal axes at P. On a plane passing through P with normal $\boldsymbol{n}$, the resultant stress vector is $\overset{n}{\boldsymbol{T}}$ and its components, according to Eq. (1.31), are

$$\overset{n}{\boldsymbol{T}}_x = \sigma_1 n_x, \qquad \overset{n}{\boldsymbol{T}}_y = \sigma_2 n_y, \qquad \overset{n}{\boldsymbol{T}}_z = \sigma_3 n_z$$

Let PQ be along the resultant stress vector and its length be equal to its magnitude, i.e. $PQ = |\overset{n}{\boldsymbol{T}}|$. The coordinates (x, y, z) of the point Q are then

$$x = \overset{n}{\boldsymbol{T}}_x, \qquad y = \overset{n}{\boldsymbol{T}}_y, \qquad z = \overset{n}{\boldsymbol{T}}_z$$

Since $n_x^2 + n_y^2 + n_z^2 = 1$, we get from the above two equations.

$$\frac{x^2}{\sigma_1^2} + \frac{y^2}{\sigma_2^2} + \frac{z^2}{\sigma_3^2} = 1 \tag{1.55}$$

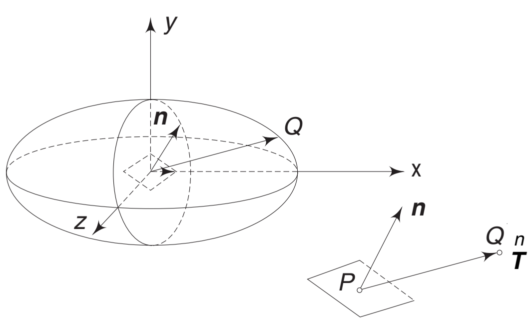

Fig. 1.22 *Lame's ellipsoid*

This is the equation of an ellipsoid referred to the principal axes. This ellipsoid is called the stress ellipsoid or Lame's ellipsoid. One of its three semi-axes is the longest, the other the shortest, and the third in-between (Fig.1.22). These are the extermum values.

If two of the principal stresses are equal, for instance

$\sigma_1 = \sigma_2$, Lame's ellipsoid is an ellipsoid of revolution and the state of stress at a given point is symmetrical with respect to the third principal axis *Pz*. If all the principal stresses are equal, $\sigma_1 = \sigma_2 = \sigma_3$, Lame's ellipsoid becomes a sphere.

Each radius vector *PQ* of the stress ellipsoid represents to a certain scale, the resultant stress on one of the planes through the centre of the ellipsoid. It can be shown (Example 1.11) that the stress represented by a radius vector of the stress ellipsoid acts on the plane parallel to tangent plane to the surface called the stress-director surface, defined by

$$\frac{x^2}{\sigma_1} + \frac{y^2}{\sigma_2} + \frac{z^2}{\sigma_3} = 1 \tag{1.56}$$

The tangent plane to the stress-director surface is drawn at the point of intersection of the surface with the radius vector. Consequently, Lame's ellipsoid and the stress-director surface together completely define the state of stress at point *P*.

Example 1.11 *Show that Lame's ellipsoid and the stress-director surface together completely define the state of stress at a point.*

Solution If σ_1, σ_2 and σ_3 are the principal stresses at a point *P*, the equation of the ellipsoid referred to principal axes is given by

$$\frac{x^2}{\sigma_1^2} + \frac{y^2}{\sigma_2^2} + \frac{z^2}{\sigma_3^2} = 1$$

The stress-director surface has the equation

$$\frac{x^2}{\sigma_1} + \frac{y^2}{\sigma_2} + \frac{z^2}{\sigma_3} = 1$$

It is known from analytical geometry that for a surface defined by $F(x, y, z) = 0$, the normal to the tangent at a point (x_0, y_0, z_0) has direction cosines proportional to $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$, evaluated at (x_0, y_0, z_0). Hence, at a point (x_0, y_0, z_0) on the stress ellipsoid, if **m** is the normal to the tangent plane (Fig.1.23), then

$$m_x = \alpha \frac{x_0}{\sigma_1}, \qquad m_y = \alpha \frac{y_0}{\sigma_2}, \qquad m_z = \alpha \frac{z_0}{\sigma_3}$$

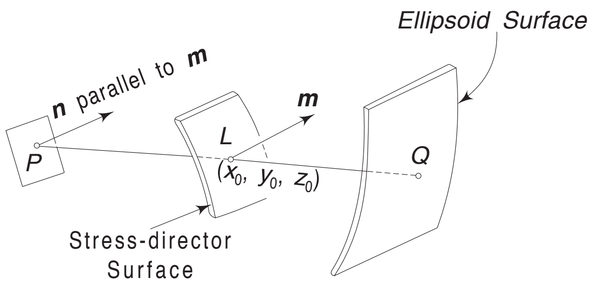

Fig. 1.23 *Stress director surface and ellipsoid surface*

Consider a plane through P with normal parallel to $\boldsymbol{m}$. On this plane, the resultant stress vector will be $\overset{n}{\boldsymbol{T}}$ with components given by

$$\overset{m}{\boldsymbol{T}}_x = \sigma_1 m_x; \qquad \overset{m}{\boldsymbol{T}}_y = \sigma_2 m_y; \qquad \overset{m}{\boldsymbol{T}}_z = \sigma_3 m_z$$

Substituting for m_x, m_y and m_z

$$\overset{m}{\boldsymbol{T}}_x = \alpha x_0, \qquad \overset{m}{\boldsymbol{T}}_y = \alpha y_0, \qquad \overset{m}{\boldsymbol{T}}_z = \alpha z_0$$

i.e. the components of stress on the plane with normal $\boldsymbol{m}$ are proportional to the coordinates (x_0, y_0, z_0). Hence the stress-director surface has the following property.

$L(x_0, y_0, z_0)$ is a point on the stress-director surface. $\boldsymbol{m}$ is the normal to the tangent plane at L. On a plane through P with normal $\boldsymbol{m}$, the resultant stress vector is $\overset{m}{\boldsymbol{T}}$ with components proportional to x_0, y_0 and z_0. This means that the components of PL are proportional to $\overset{m}{\boldsymbol{T}}_x$, $\overset{m}{\boldsymbol{T}}_y$ and $\overset{m}{\boldsymbol{T}}_z$.

PQ being an extension of PL and equal to $\overset{n}{\boldsymbol{T}}$ in magnitude, the plane having this resultant stress will have $\boldsymbol{m}$ as its normal.

1.25 THE PLANE STATE OF STRESS

If in a given state of stress, there exists a coordinate system $Oxyz$ such that for this system

$$\sigma_z = 0, \qquad \tau_{xz} = 0, \qquad \tau_{yz} = 0 \tag{1.57}$$

then the state is said to have a 'plane state of stress' parallel to the xy plane. This state is also generally known as a two-dimensional state of stress. All the foregoing discussions can be applied and the equations reduce to simpler forms as a result of Eq. (1.57). The state of stress is shown in Fig. 1.24.

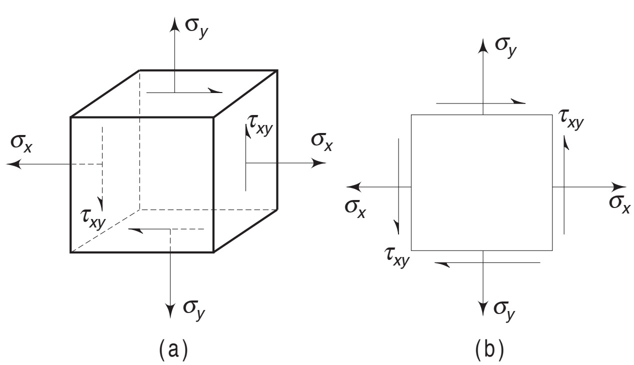

Fig. 1.24 *(a) Plane state of stress (b) Conventional representation*

Consider a plane with the normal lying in the xy plane. If n_x, n_y and n_z are the direction cosines of the normal, we have $n_x = \cos\theta$, $n_y = \sin\theta$ and $n_z = 0$ (Fig. 1.25). From Eq. (1.9)

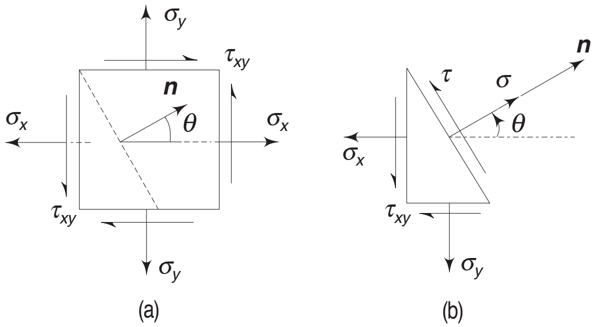

Fig. 1.25 *Normal and shear stress components on an oblique plane*

$$\overset{n}{T}_x = \sigma_x \cos\theta + \tau_{xy} \sin\theta$$
$$\overset{n}{T}_y = \sigma_y \sin\theta + \tau_{xy} \cos\theta \qquad (1.58)$$
$$\overset{n}{T}_z = 0$$

The normal and shear stress components on this plane are from Eqs (1.11a) and (1.11b)

$$\sigma = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy} \sin\theta \cos\theta$$
$$= \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta \qquad (1.59)$$

and $$\tau^2 = \overset{n}{T}{}_x^2 + \overset{n}{T}{}_y^2 - \sigma^2$$

or $$\tau = \frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta \qquad (1.60)$$

The principal stresses are given by Eq. (1.29) as

$$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \left[\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2\right]^{1/2} \qquad (1.61)$$
$$\sigma_3 = 0$$

The principal planes are given by
(i) the z plane on which $\sigma_3 = \sigma_z = 0$ and
(ii) two planes with normals in the xy plane such that

$$\tan 2\phi = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \qquad (1.62)$$

The above equation gives two planes at right angles to each other.

If the principal stresses σ_1, σ_2 and σ_3 are arranged such that $\sigma_1 \geq \sigma_2 \geq \sigma_3$, the maximum shear stress at the point will be

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} \qquad (1.63a)$$

In the xy plane, the maximum shear stress will be

$$\tau_{\max} = \frac{1}{2}(\sigma_1 - \sigma_2) \tag{1.63b}$$

and from Eq. (1.61)

$$\tau_{\max} = \left[\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2\right]^{1/2} \tag{1.64}$$

1.26 DIFFERENTIAL EQUATIONS OF EQUILIBRIUM

So far, attention has been focussed on the state of stress at a point. In general, the state of stress in a body varies from point to point. One of the fundamental problems in a book of this kind is the determination of the state of stress at every point or at any desired point in a body. One of the important sets of equations used in the analyses of such problems deals with the conditions to be satisfied by the stress components when they vary from point to point. These conditions will be established when the body (and, therefore, every part of it) is in equillibrium. We isolate a small element of the body and derive the equations of equilibrium from its free-body diagram (Fig. 1.26). A similar procedure was adopted in Sec. 1.8 for establishing the equality of cross shears.

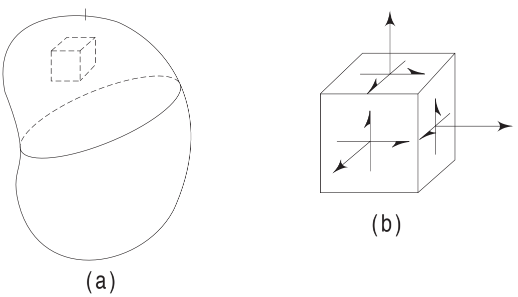

Fig. 1.26 *Isolated cubical element in equilibrium*

Consider a small rectangular element with sides Δx, Δy and Δz isolated from its parent body. Since in the limit, we are going to make Δx, y and Δz tend to zero, we shall deal with average values of the stress components on each face. These stress components are shown in Fig. 1.27.

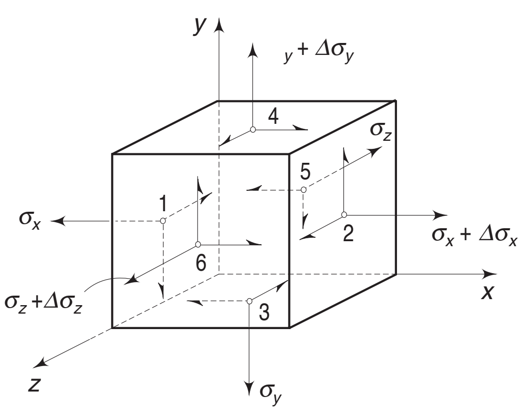

Fig. 1.27 *Variation of stresses*

The faces are marked as 1, 2, 3 etc. On the left hand face, i.e. face No. 1, the average stress components are σ_x, τ_{xy} and τ_{xz}. On the right hand face, i.e. face No. 2, the average stress components are

$$\sigma_x + \frac{\partial \sigma_x}{\partial x}\Delta x, \qquad \tau_{xy} + \frac{\partial \tau_{xy}}{\partial x}\Delta x, \qquad \tau_{xz} + \frac{\partial \tau_{xz}}{\partial x}\Delta x$$

This is because the right hand face is Δx distance away from the left hand face. Following a similar procedure, the stress components on the six faces of the element are as follows:

Face 1 $\quad \sigma_x, \quad \tau_{xy}, \quad \tau_{xz}$

Face 2 $\quad \sigma_x + \dfrac{\partial \sigma_x}{\partial x}\Delta x, \quad \tau_{xy} + \dfrac{\partial \tau_{xy}}{\partial x}\Delta x, \quad \tau_{xz} + \dfrac{\partial \tau_{xz}}{\partial x}\Delta x$

Face 3 $\quad \sigma_y, \quad \tau_{yx}, \quad \tau_{yz}$

Face 4

$$\sigma_y + \frac{\partial \sigma_y}{\partial y}\Delta y, \quad \tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\Delta y, \quad \tau_{yz} + \frac{\partial \tau_{yz}}{\partial y}\Delta y$$

Face 5 $\quad \sigma_z, \quad \tau_{zx}, \quad \tau_{zy}$

Face 6 $\quad \sigma_z + \dfrac{\partial \sigma_z}{\partial z}\Delta z, \quad \tau_{zx} + \dfrac{\partial \tau_{zx}}{\partial z}\Delta z, \quad \tau_{zy} + \dfrac{\partial \tau_{zy}}{\partial z}\Delta z$

Let the body force components per unit volume in the x, y and z directions be γ_x, γ_y, and γ_z. For equilibrium in x direction

$$\left(\sigma_x + \frac{\partial \sigma_x}{\partial x}\Delta x\right)\Delta y\,\Delta z - \sigma_x\,\Delta y\,\Delta z + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\Delta y\right)\Delta z\,\Delta x - \tau_{yx}\,\Delta z\,\Delta x +$$

$$\left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\Delta z\right)\Delta x\,\Delta y - \tau_{zx}\,\Delta x\,\Delta y + \gamma_x\,\Delta x\,\Delta y\,\Delta z = 0$$

Cancelling terms, dividing by Δx, Δy, Δz and going to the limit, we get

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \gamma_x = 0$$

Similarly, equating forces in the y and z directions respectively to zero, we get two more equations. On the basis of the fact that the cross shears are equal, i.e. $\tau_{xy} = \tau_{yx}$, $\tau_{yz} = \tau_{zy}$, $\tau_{xz} = \tau_{zx}$, we obtain the three differential equations of equilibrium as

$$\begin{aligned}
&\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \gamma_x = 0\\
&\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} + \gamma_y = 0\\
&\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \gamma_z = 0
\end{aligned} \tag{1.65}$$

Equations (1.65) must be satisfied at all points throughout the volume of the body. It must be recalled that the moment equilibrium conditions established the equality of cross shears in Sec.1.8.

1.27 EQUILIBRIUM EQUATIONS FOR PLANE STRESS STATE

The plane stress has already been defined. If there exists a plane stress state in the xy plane, then $\sigma_z = \tau_{zx} = \tau_{yz} = \gamma_z = 0$ and only σ_x, σ_y, τ_{xy}, γ_x and γ_y exist. The differnetial equations of equilibrium become

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \gamma_x = 0$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \gamma_y = 0 \tag{1.66}$$

Example 1.12 *The cross-section of the wall of a dam is shown in Fig. 1.28. The pressure of water on face OB is also shown. With the axes Ox and Oy, as shown in Fig. 1.28, the stresses at any point (x, y) are given by (γ = specific weight of water and ρ = specific weight of dam material)*

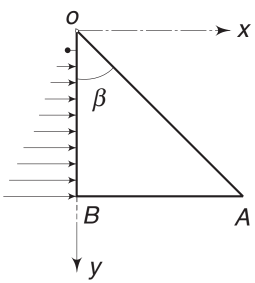

Fig 1.28 *Example 1.12*

$$\sigma_x = -\gamma y$$

$$\sigma_y = \left(\frac{\rho}{\tan\beta} - \frac{2\gamma}{\tan^3\beta}\right)x + \left(\frac{\gamma}{\tan^2\beta} - \rho\right)y$$

$$\tau_{xy} = \tau_{yx} = -\frac{\gamma}{\tan^2\beta}x$$

$$\tau_{yz} = 0, \quad \tau_{zx} = 0, \quad \sigma_z = 0$$

Check if these stress components satisfy the differential equations of equilibrium. Also, verify if the boundary conditions are satisfied on face OB.

Solution The equations of equilibrium are

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \gamma_x = 0$$

and

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \gamma_y = 0$$

Substituting and noting that $\gamma_x = 0$ and $\gamma_y = \rho$, the first equation is satisifed. For the second equation also

$$\frac{\gamma}{\tan^2\beta} - \rho - \frac{\gamma}{\tan^2\beta} + \rho = 0$$

On face OB, at any y, the stress components are $\sigma_x = -\gamma y$ and $\tau_{xy} = 0$. Hence the boundary conditions are also satisfied.

Example 1.13 *Consider a function $\phi(x, y)$, which is called the stress function. If the values of σ_x, σ_y and τ_{xy} are as given below, show that these satisfy the differential equations of equilibrium in the absence of body forces.*

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2}, \qquad \sigma_y = \frac{\partial^2 \phi}{\partial x^2}, \qquad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x\, \partial y}$$

Solution Substituting in the differential equations of equilibrium

$$\frac{\partial^3 \phi}{\partial y^2\, \partial x} - \frac{\partial^3 \phi}{\partial y^2\, \partial x} = 0$$

$$\frac{\partial^3 \phi}{\partial x^2\, \partial y} - \frac{\partial^3 \phi}{\partial x^2\, \partial y} = 0$$

Example 1.14 *Consider the rectangular beam shown in Fig. 1.29. According to the elementary theory of bending, the 'fibre stress' in the elastic range due to bending is given by*

$$\sigma_x = -\frac{My}{I} = -\frac{12\, My}{bh^3}$$

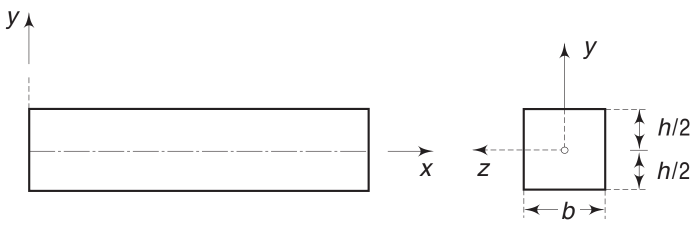

Fig. 1.29 *Exmaple 1.14*

where M is the bending moment which is a function of x. Assume that $\sigma_z = \tau_{zx} = \tau_{zy} = 0$ and also that $\tau_{xy} = 0$ at the top and bottom, and further, that $\sigma_y = 0$ at the bottom. Using the differential equations of equilibrium, determine τ_{xy} and σ_y. Compare these with the values given in the elementary strength of materials.

Solution From Eq. (1.65)

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0$$

Since $\tau_{xz} = 0$ and M is a function of x

$$-\frac{12y}{bh^3}\frac{\partial M}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0$$

or $$\frac{\partial \tau_{xy}}{\partial y} = \frac{12}{bh^3}\frac{\partial M}{\partial x}\, y$$

Integrating $$\tau_{xy} = \frac{6}{bh^3}\frac{\partial M}{\partial x}\, y^2 + c_1 f(x) + c_2$$

where $f(x)$ is a function of x alone and c_1, c_2 are constants. It is given that $\tau_{xy} = 0$ at $y = \pm\dfrac{h}{2}$

$$\therefore \qquad \frac{6}{bh^3}\frac{h^2}{4}\frac{\partial M}{\partial x} = -c_1 f(x) - c_2$$

or
$$c_1 f(x) + c_2 = -\frac{3}{2bh}\frac{\partial M}{\partial x}$$

$$\therefore \qquad \tau_{xy} = \frac{3}{2bh}\frac{\partial M}{\partial x}\left(\frac{4y^2}{h^2} - 1\right)$$

From elementary strength of materials, we have

$$\tau_{xy} = \frac{V}{Ib}\int_y^{h/2} y'\, dA$$

where $V = -\dfrac{\partial M}{\partial x}$ is the shear force. Simplifying the above expression

$$\tau_{xy} = -\frac{\partial M}{\partial x}\frac{12}{b^2\, h^3}\left(\frac{h^2}{4} - y^2\right)\frac{b}{2}$$

or
$$\tau_{xy} = \frac{3}{2bh}\frac{\partial M}{\partial x}\left(\frac{4y^2}{h^3} - 1\right)$$

i.e. the same as the expression obtained above.

From the next equilibrium equation, i.e. from

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} = 0$$

we get
$$\frac{\partial \sigma_y}{\partial y} = -\frac{3}{2bh}\left(\frac{4y^2}{h^2} - 1\right)\frac{\partial^2 M}{\partial x^2}$$

$$\therefore \qquad \sigma_y = -\frac{3}{2bh}\frac{\partial^2 M}{\partial x^2}\left(\frac{4y^3}{3h^2} - y\right) + c_3\, F(x) + c_4$$

where $F(x)$ is a function of x alone. It is given that $\sigma_y = 0$ at $y = -\dfrac{h}{2}$.

Hence,
$$c_3 F(x) + c_4 = \frac{3}{2bh}\frac{\partial^2 M}{\partial x^2}\frac{h}{3}$$

$$= \frac{1}{2b}\frac{\partial^2 M}{\partial x^2}$$

Substituting

$$\sigma_y = -\frac{3}{2bh}\frac{\partial^2 M}{\partial x^2}\left(\frac{4y^3}{3h^2} - y - \frac{h}{3}\right)$$

At $y = +h/2$, the value of σ_y is

$$\sigma_y = \frac{1}{b}\frac{\partial^2 M}{\partial x^2} = \frac{w}{b}$$

where w is the intensity of loading. Since b is the width of the beam, the stress will be w/b as obtained above.

1.28 BOUNDARY CONDITIONS

Equation (1.66) must be satisfied throughout the volume of the body. When the stresses vary over the plate (i.e. the body having the plane stress state), the stress components σ_x, σ_y and τ_{xy} must be consistent with the externally applied forces at a boundary point.

Consider the two-dimensional body shown in Fig.1.30. At a boundary point P, the outward drawn normal is $\boldsymbol{n}$. Let F_x and F_y be the components of the surface forces per unit area at this point.

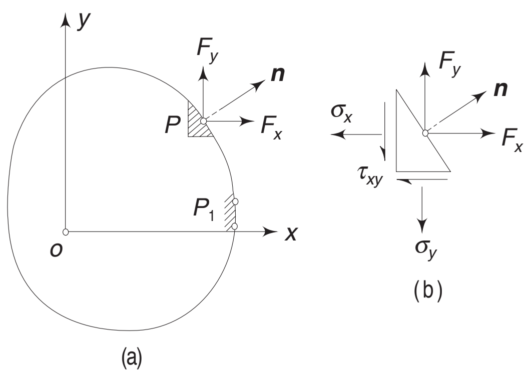

Fig. 1.30 *(a) Element near a boundary point (b) Free body diagram*

F_x and F_y must be continuations of the stresses σ_x, σ_y and τ_{xy} at the boundary. Hence, using Cauchy's equations

$$\overset{n}{\boldsymbol{T}}_x = F_x = \sigma_x n_x + \tau_{xy} n_y$$

$$\overset{n}{\boldsymbol{T}}_y = F_y = \sigma_y n_y + \tau_{xy} n_x$$

If the boundary of the plate happens to be parallel to y axis, as at point P_1, the boundary conditions become

$$F_x = \sigma_x \quad \text{and} \quad F_y = \tau_{xy}$$

1.29 EQUATIONS OF EQUILIBRIUM IN CYLINDRICAL COORDINATES

Till this section, we have been using a rectangular or the Cartesian frame of reference for analyses. Such a frame of reference is useful if the body under analysis happens to possess rectangular or straight boundaries. Numerous problems exist where the bodies under discussion possess radial symmetry; for example, a thick cylinder subjected to internal or external pressure. For the analysis of such problems, it is more convenient to use polar or cylindrical coordinates. In this section, we shall develop some equations in cylindrical coordinates.

Consider an axisymmetric body as shown in Fig. 1.31(a). The axis of the body is usually taken as the z axis. The two other coordinates are r and θ, where θ is measured counter-clockwise. The rectangular stress components at a point $P(r, \theta, z)$ are

$$\sigma_r, \sigma_\theta, \sigma_z, \tau_{\theta r}, \tau_{\theta z} \text{ and } \tau_{zr}$$

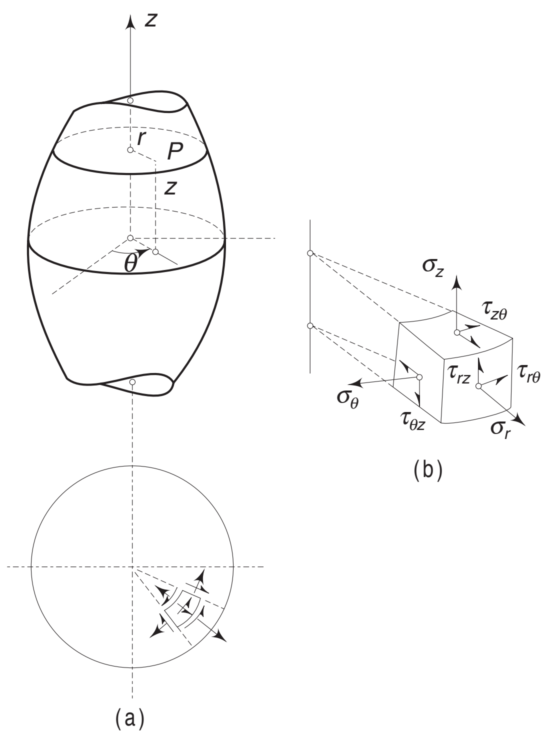

Fig. 1.31 *(a) Cylindrical coordinates of a point (b) Stresses on an element*

These are shown acting on the faces of a radial element at point P in Fig.1.31(b). σ_r, σ_θ and σ_z are called the radial, circumferential and axial stresses respectively. If the stresses vary from point to point, one can derive the appropriate differential equations of equilibrium, as in Sec. 1.26. For this purpose, consider a cylindrical element having a radial length Δr with an included angle $\Delta\theta$ and a height Δz, isolated from the body. The free-body diagram of the element is shown in Fig.1.32(b). Since the element is very small, we work with the average stresses acting on each face.

The area of the face $aa'd'd$ is $r\,\Delta\theta\,\Delta z$ and the area of face $bb'c'c$ is $(r + \Delta r)\,\Delta\theta\,\Delta z$. The areas of faces $dcc'd'$ and $abb'e'$ are each equal to $\Delta r\,\Delta z$.

The faces $abcd$ and $a'b'c'd'$ have each an area $\left(r + \dfrac{\Delta r}{2}\right)\Delta\theta\,\Delta r$. The average stresses on these faces (which are assumed to be acting at the mid point of eace face) are

On face $aa'd'd$

normal stress σ_r

tangential stresses τ_{rz} and $\tau_{r\theta}$

On face $bb'c'c$

normal stress $\sigma_r + \dfrac{\partial \sigma_r}{\partial r}\Delta r$

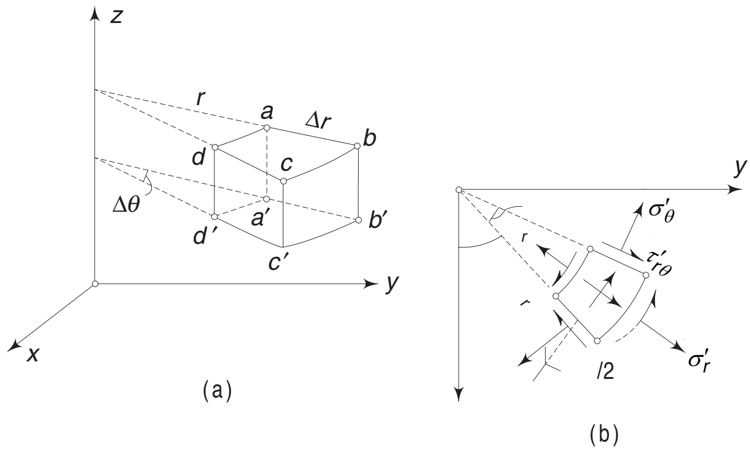

Fig. 1.32 *(a) Geometry of cylindrical element (b) Variation of stresses across faces*

tangential stresses $\tau_{rz} + \dfrac{\partial \tau_{rz}}{\partial r}\Delta r$ and $\tau_{r\theta} + \dfrac{\partial \tau_{\theta r}}{\partial r}\Delta r$

The changes are because the face $bb'c'c$ is Δr distance away from the face $aa'd'd$.

On face $dcc'd'$

normal stress σ_θ

tangential stresses $\tau_{r\theta}$ and $\tau_{\theta z}$

On face $abb'a$

normal stress $\sigma_\theta + \dfrac{\partial \sigma_\theta}{\partial \theta}\Delta\theta$

tangential stresses $\tau_{r\theta} + \dfrac{\partial \tau_{r\theta}}{\partial \theta}\Delta\theta$ and $\tau_{\theta z} + \dfrac{\partial \tau_{\theta z}}{\partial \theta}\Delta\theta$

The changes in the above components are because the face $abb'a$ is separated by an angle $\Delta\theta$ from the face $dcc'd'$.

On face $a'b'c'd'$

normal stress σ_z

tangential stresses τ_{rz} and $\tau_{\theta z}$

On face $abcd$

normal stress $\sigma_z + \dfrac{\partial \sigma_z}{\partial z}\Delta z$

tangential stresses $\tau_{rz} + \dfrac{\partial \tau_{rz}}{\partial z}\Delta z$ and $\tau_{\theta z} + \dfrac{\partial \tau_{\theta z}}{\partial z}\Delta z$

Let γ_r, γ_θ and γ_z be the body force components per unit volume. If the element is in equilibrium, the sum of forces in r, θ and z directions must vanish individually, Equating the forces in r direction to zero,

$$\left(\sigma_r + \frac{\partial \sigma_r}{\partial r}\Delta r\right)(r + \Delta r)\,\Delta\theta\Delta z + \left(\tau_{rz} + \frac{\partial \tau_{rz}}{\partial z}\Delta z\right)\left(r + \frac{\Delta r}{2}\right)\Delta\theta\Delta r$$

$$- \sigma_r \, r \, \Delta\theta \, \Delta z - \tau_{rz} \left(r + \frac{\Delta r}{2} \right) \Delta\theta \, \Delta r - \sigma_\theta \sin \frac{\Delta\theta}{2} \Delta r \, \Delta z$$

$$- \tau_{r\theta} \cos \frac{\Delta\theta}{2} \Delta r \, \Delta z - \left(\sigma_\theta + \frac{\partial \sigma_\theta}{\partial \theta} \Delta\theta \right) \sin \frac{\Delta\theta}{2} \Delta r \, \Delta z$$

$$+ \left(\tau_{r\theta} + \frac{\partial \tau_{r\theta}}{\partial \theta} \Delta\theta \right) \cos \frac{\Delta\theta}{2} \Delta r \, \Delta z + \gamma_r \left(r + \frac{\Delta r}{2} \right) \Delta\theta \, \Delta r \, \Delta z = 0$$

Cancelling terms, dividing by $\Delta\theta \, \Delta r \, \Delta z$ and going to the limit with $\Delta\theta$, Δr and Δz, all tending to zero

$$r \frac{\partial \sigma_r}{\partial r} + r \frac{\partial \tau_{rz}}{\partial z} + \frac{\partial \tau_{r\theta}}{\partial \theta} + \sigma_r - \sigma_\theta + r \gamma_r = 0$$

or

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} + \gamma_r = 0 \tag{1.67}$$

Similarly, for equilibrium in z and θ directions, we get

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{1}{r} \frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\tau_{rz}}{r} + \gamma_z = 0 \tag{1.68}$$

and

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{1}{r} \frac{\partial \sigma_\theta}{\partial \theta} + \frac{2 \, \tau_{r\theta}}{r} + \gamma_\theta = 0 \tag{1.69}$$

Equations (1.67)–(1.69) are the differential equations of equilibrium expressed in polar coordinates.

1.30 AXISYMMETRIC CASE AND PLANE STRESS CASE

If an axisymmetric body is loaded symmetrically, the stress components do not depend on θ. Since the deformations are symmetric, $\tau_{r\theta}$ and $\tau_{\theta z}$ do not exist and consequently the above set of equations in the absence of body forces are reduced to

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} = 0$$

A sphere under diametral compression or a cone under a load at the apex are examples to which the above set of equations can be applied.

If the state of stress is two-dimensional in nature, i.e. plane stress state, then only σ_r, σ_θ, $\tau_{r\theta}$, γ_r, and γ_θ exist. The other stress components vanish.These non-vanishing stress components depend only on θ and r and are independent of z in the absence of body forces. The equations of equilibrium reduce to

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_\theta}{\partial \theta} + \frac{2\tau_{r\theta}}{r} = 0 \tag{1.70}$$

Example 1.15 *Consider a function $\phi(r, \theta)$, which is called the stress function. If the values of σ_r, σ_θ, and $\tau_{r\theta}$ are as given below, show that in the absence of body forces, these satisfy the differential equations of equilibrium.*

$$\sigma_r = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}$$

$$\sigma_\theta = \frac{\partial^2 \phi}{\partial r^2}$$

$$\tau_{r\theta} = -\frac{1}{r}\frac{\partial^2 \phi}{\partial r\,\partial \theta} + \frac{1}{r^2}\frac{\partial \phi}{\partial \theta}$$

Solution The equations of equilibrium are

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{2\tau_{r\theta}}{r} = 0$$

Substituting the stress function in the first equation of equilibrium,

$$-\frac{1}{r^2}\frac{\partial \phi}{\partial r} + \frac{1}{r}\frac{\partial^2 \phi}{\partial r^2} - \frac{2}{r^3}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{r^2}\frac{\partial^3 \phi}{\partial \theta^2\,\partial r} + \frac{1}{r}\left(-\frac{1}{r}\frac{\partial^3 \phi}{\partial \theta^2\,\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}\right)$$

$$+\frac{1}{r^2}\frac{\partial \phi}{\partial r} + \frac{1}{r^3}\frac{\partial^2 \phi}{\partial \theta^2} - \frac{1}{r}\frac{\partial^2 \phi}{\partial r^2} = 0$$

Hence, the first equation is satisfied. Similarly, it can easily be verified that the second condition also holds good.

Problems

1.1 It was assumed in Sec.1.2 that across any infinitesimal surface element in a solid, the action of the exterior material upon the interior is equipollent (i.e. equal in strength or effect) to only a force. It is also possible to assume that in addition to a force, there is also a couple, i.e. at any point across any plane ***n***, there is a stress vector $\overset{n}{\boldsymbol{T}}$ and a couple-stress vector $\overset{n}{\boldsymbol{M}}$, as shown in Fig. 1.33.

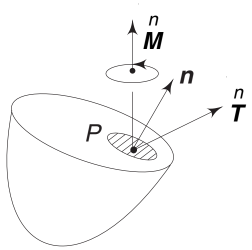

Fig. 1.33 *Problem 1.1*

Show that a set of equations similar to Cauchy's equations can be derived, i.e. if we know the couple-stress vectors on three mutually perpendicular planes passing through the point *P*, then we can determine the couple-stress vector on any plane ***n*** passing through the point. The equations are

$$\overset{n}{\boldsymbol{M}}_x = M_{xx}\, n_x + M_{yx}\, n_y + M_{zx}\, n_z$$

$$\overset{n}{\boldsymbol{M}}_y = M_{xy}\, n_x + M_{yy}\, n_y + M_{zy}\, n_z$$

$$\overset{n}{\boldsymbol{M}}_z = M_{xz}\, n_x + M_{yz}\, n_y + M_{zz}\, n_z$$

$\overset{n}{\boldsymbol{M}}_x$, $\overset{n}{\boldsymbol{M}}_y$, $\overset{n}{\boldsymbol{M}}_z$ are the x, y and z components of the vector $\overset{n}{\boldsymbol{M}}$ acting on plane $\boldsymbol{n}$.

1.2 A rectangular beam is subjected to a pure bending moment M. The cross-section of the beam is shown in Fig. 1.34. Using the elementary flexure formula, determine the normal and shearing stresses at a point (x, y) on the plane AB shown.

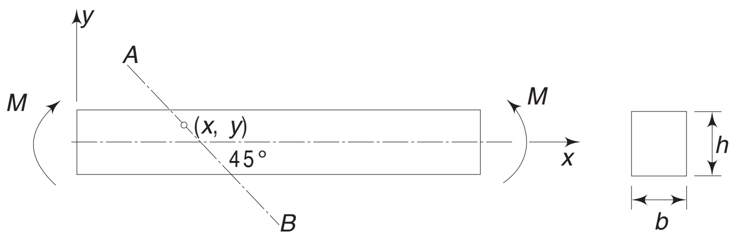

Fig. 1.34 *Problem 1.2*

$$\left[\textit{Ans.}\quad \sigma_n = \tau_n = \frac{6My}{bh^3}\right]$$

1.3 Consider a sphere of radius R subjected to diametral compression (Fig. 1.35). Let σ_r, σ_θ and σ_ϕ be the normal stresses and $\tau_{r\theta}$, $\tau_{\theta\phi}$ and $\tau_{\phi r}$ the shear stresses at a point. At point $P(o, y, z)$ on the surface and lying in the yz plane, determine the rectangular normal stress components σ_x, σ_y and σ_z in terms of the spherical stress components.

[*Ans.* $\sigma_x = \sigma_\theta$; $\sigma_y = \sigma_\phi \cos^2\phi$; $\sigma_z = \sigma_\phi \sin^2\phi$]

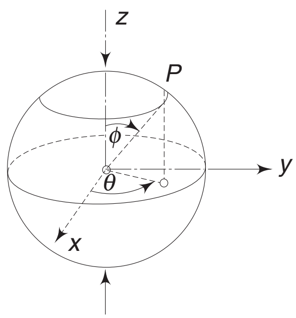

Fig. 1.35 *Problem 1.3*

1.4 The state of stress at a point is characterised by the matrix shown. Determine T_{11} such that there is at least one plane passing through the point in such a way that the resultant stress on that plane is zero. Determine the direction cosines of the normal to that plane.

$$\left[\tau_{ij}\right] = \begin{bmatrix} T_{11} & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

$$\left[\textit{Ans.} \quad T_{11} = 2;\, n_x = \pm\frac{2}{3};\, n_y = \pm\frac{1}{3};\, n_z = \pm\frac{2}{3} \right]$$

1.5 If the rectangular components of stress at a point are as in the matrix below, determine the unit normal of a plane parallel to the z axis, i.e. $n_z = 0$, on which the resultant stress vector is tangential to the plane

$$\left[\tau_{ij}\right] = \begin{bmatrix} a & 0 & d \\ 0 & b & e \\ d & e & c \end{bmatrix}$$

$$\left[\textit{Ans.} \quad n_x = \pm\left(\frac{b}{b-a}\right)^{1/2};\, n_y = \pm\left(\frac{a}{a-b}\right)^{1/2};\, n_z = 0 \right]$$

1.6 A cross-section of the wall of a dam is shown in Fig.1.36. The pressure of water on face OB is also shown. The stresses at any point (x, y) are given by the following expressions

$$\sigma_x = -\gamma y$$

$$\sigma_y = \left(\frac{\rho}{\tan\beta} - \frac{2\gamma}{\tan^3\beta}\right)x + \left(\frac{\gamma}{\tan^2\beta} - \rho\right)y$$

$$\tau_{xy} = \tau_{yx} = -\frac{\gamma x}{\tan^2\beta}$$

$$\tau_{yz} = \tau_{zx} = \sigma_z = 0$$

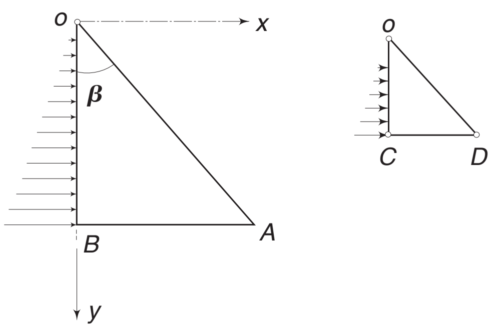

Fig. 1.36 *Problem 1.6*

where γ is the specific weight of water and ρ the specific weight of the dam material.

Consider an element OCD and show that this element is in equilibrium under the action of the external forces (water pressure and gravity force) and the internally distributed forces across the section CD.

1.7 Determine the principal stresses and their axes for the states of stress characterised by the following stress matrices (units are 1000 kPa).

(i) $\left[\tau_{ij}\right] = \begin{bmatrix} 18 & 0 & 24 \\ 0 & -50 & 0 \\ 24 & 0 & 32 \end{bmatrix}$

$$\left[\begin{array}{l} \textit{Ans. } \sigma_1 = 50, n_x = 0.6, n_y = 0, n_z = 0.8 \\ \sigma_2 = 0, n_x = 0.8, n_y = 0, n_z = 0.6 \\ \sigma_3 = -50, n_x = n_z = 0, n_y = 1 \end{array}\right]$$

(ii) $\left[\tau_{ij}\right] = \begin{bmatrix} 3 & -10 & 0 \\ -10 & 0 & 30 \\ 0 & 30 & -27 \end{bmatrix}$

$$\left[\begin{array}{l} \textit{Ans. } \sigma_1 = 23, n_x = 0.394, n_y = 0.788, n_z = 0.473 \\ \sigma_2 = 0, n_x = 0.912, n_y = 0.274, n_z = 0.304 \\ \sigma_3 = -47, n_x = 0.941, n_y = 0.188, n_z = 0.288 \end{array}\right]$$

1.8 The state of stress at a point is characterised by the components

$$\sigma_x = 12.31, \quad \sigma_y = 8.96, \quad \sigma_z = 4.34$$

$$\tau_{xy} = 4.20, \quad \tau_{yx} = 5.27, \quad \sigma_z = 0.84$$

Find the values of the principal stresses and their directions

$$\left[\begin{array}{l} \textit{Ans. } \sigma_1 = 16.41, n_x = 0.709, n_y = 0.627, n_z = 0.322 \\ \sigma_2 = 8.55, n_x = 0.616, n_y = 0.643, n_z = 0.455 \\ \sigma_3 = 0.65, n_x = 0.153, n_y = 0.583, n_z = 0.798 \end{array}\right]$$

1.9 For Problem 1.8, determine the principal shears and the associated normal stresses.

$$\left[\begin{array}{l} \textit{Ans. } \tau_3 = 3.94, \sigma_n = 12.48 \\ \tau_2 = 7.88, \sigma_n = 8.53 \\ \tau_1 = 3.95, \sigma_n = 4.52 \end{array}\right]$$

1.10 For the state of stress at a point characterised by the components (in 1000 kPa)

$$\sigma_x = 12, \quad \sigma_y = 4, \quad \sigma_z = 10, \quad \tau_{xy} = 3, \quad \tau_{yz} = \tau_{zx} = 0$$

determine the principal stresses and their directions.

$$\left[\begin{array}{l} \textit{Ans. } \sigma_1 = 13;\ 18^\circ \text{ with } x \text{ axis};\ n_z = 0 \\ \sigma_2 = 10;\ n_x = 0;\ n_y = 0;\ n_z = 1 \\ \sigma_3 = 3;\ -72^\circ \text{ with } x \text{ axis};\ n_z = 0 \end{array}\right]$$

1.11 Let $\sigma_x = -5c$, $\sigma_y = c$, $\sigma_z = c$, $\tau_{xy} = -c$, $\tau_{yz} = \tau_{zx} = 0$, where $c = 1000$ kPa. Determine the principal stresses, stress deviators, principal axes, greatest shearing stress and octahedral stresses.

$$\left[\begin{array}{l} \textit{Ans. } \sigma_1 = (-2 + \sqrt{10})c;\ n_z = 0 \text{ and } \theta = 9.2^\circ \text{ with } y \text{ axis} \\ \sigma_2 = c,\ n_x = n_y = 0;\ n_z = 1 \\ \sigma_3 = (-2 - \sqrt{10})c;\ n_z = 0 \text{ and } \theta = 9.2^\circ \text{ with } x \text{ axis} \\ \tau_{max} = \sqrt{10}\,c;\ \sigma'_x = -4c;\ \sigma'_y = 2c;\ \sigma'_z = 2c \\ \sigma_{oct} = -c;\ \tau_{oct} = \dfrac{\sqrt{78}}{3}c \end{array}\right]$$

1.12 A solid shaft of diameter $d = \sqrt{10}$ cm (Fig. 1.37) is subjected to a tensile force P = 10,000 N and a torque T = 5000 N cm. At point A on the surface, determine the principal stresses, the octahedral shearing stress and the maximum shearing stress.

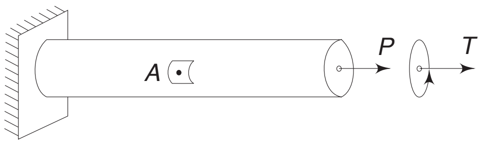

Fig. 1.37 *Problem 1.12*

$$\left[\begin{array}{l} \textit{Ans.} \quad \sigma_{1,2} = \dfrac{2000}{\pi}\left(1 \pm \sqrt{13/5}\right) Pa \\ \tau_{\max} = \dfrac{2000}{\pi}\sqrt{\dfrac{13}{5}}\, Pa \\ \tau_{oct} = \dfrac{4000}{3\pi}\sqrt{\dfrac{22}{5}}\, Pa \end{array}\right]$$

1.13 A cylindrical rod (Fig. 1.38) is subjected to a torque T. At any point P of the cross-section LN, the following stresses occur

$$\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = \tau_{yx} = 0,\ \tau_{xz} = \tau_{zx} = -G\theta y,\ \tau_{yz} = \tau_{zy} = G\theta x$$

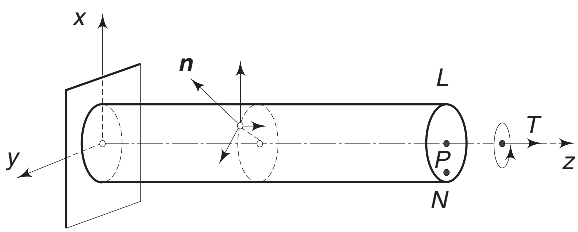

Fig. 1.38 *Problem 1.13*

Check whether these satisfy the equations of equilibrium. Also show that the lateral surface is free of load, i e. show that

$$\overset{n}{\boldsymbol{T}}_x = \overset{n}{\boldsymbol{T}}_y = \overset{n}{\boldsymbol{T}}_z = 0$$

1.14 For the state of stress given in Problem 1.13, determine the principal shears, octahedral shear stress and its associated normal stress.

$$\left[\begin{array}{l} \textit{Ans.} \quad \tau_1 = \tau_3 = \dfrac{1}{2}\, G\theta\sqrt{x^2 + y^2}\ ;\ \tau_2 = -G\theta\sqrt{x^2 + y^2} \\ \tau_{\text{oct}} = \dfrac{\sqrt{6}}{3}\, G\theta\left(\sqrt{x^2 + y^2}\right);\ \sigma_{\text{oct}} = 0 \end{array}\right]$$

Appendix 1

Mohr's Circles

It was stated in Sec. 1.17 that when points with coordinates (σ, τ) for all possible planes passing through a point are marked on the σ–τ plane, as in Fig. 1.16, the points are bounded by the three Mohr's circles. The same equations can be used to determine graphically the normal and shearing stresses on any plane with normal $\boldsymbol{n}$. Equations (1.40)–(1.42) of Sec.1.18 are

$$n_x^2 = \frac{(\sigma - \sigma_2)(\sigma - \sigma_3) + \tau^2}{(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3)} \tag{A1.1}$$

$$n_y^2 = \frac{(\sigma - \sigma_3)(\sigma - \sigma_1) + \tau^2}{(\sigma_2 - \sigma_3)(\sigma_2 - \sigma_1)} \tag{A1.2}$$

$$n_z^2 = \frac{(\sigma - \sigma_1)(\sigma - \sigma_2) + \tau^2}{(\sigma_3 - \sigma_1)(\sigma_3 - \sigma_2)} \tag{A1.3}$$

For the above equations, the principal axes coincide with the coordinate axes x, y and z. Construct a sphere of unit radius with P as the centre. P_1, P_2 and P_3 are the poles of this sphere (Fig.A1.1). Consider a point N on the surface of the sphere. The radius vector PN makes angles α, β and γ, respectively with the x, y and z axes. A plane through P with PN as normal will be parallel to a tangent plane at N to the unit sphere. If n_x, n_y and n_z are the direction cosines of the normal $\boldsymbol{n}$ to such a plane through P, then $n_x = \cos\alpha$, $n_y = \cos\beta$, $n_z = \cos\gamma$.

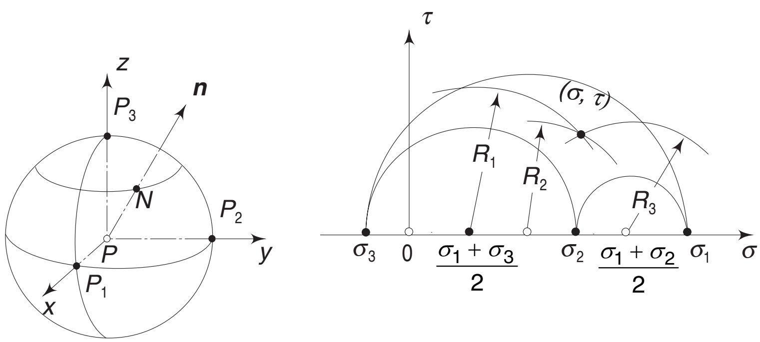

Fig. A1.1 *Mohr's circles for three-dimensional state of stress*

Let point N move in such a manner that γ remains constant. This gives a circle parallel to the equatorial circle P_1P_2.

From Eq. (A1.3)

$$(\sigma - \sigma_1)(\sigma - \sigma_2) + \tau^2 = n_z^2\ (\sigma_3 - \sigma_1)(\sigma_3 - \sigma_2)$$

or $$\left(\sigma - \frac{\sigma_1 + \sigma_2}{2}\right)^2 + \tau^2 = n_z^2\ (\sigma_3 - \sigma_1)(\sigma_3 - \sigma_2) + \frac{(\sigma_1 - \sigma_2)^2}{4} = R_3^2$$

Since $n_z = \cos\gamma$ is a constant, the above equation describes a circle in the $\sigma - \tau$ plane with the centre at $\frac{\sigma_1 + \sigma_2}{2}$ on the σ axis and radius equal to R_3. This circle gives the values of σ and τ as N moves with γ constant. For different values of n_z, one gets a family of circles, all with centres at $\frac{\sigma_1 + \sigma_2}{2}$. If $n_z = 0$ we get a Mohr's circle.

Similarly, if $n_y = \cos\beta$ is kept constant, the point N on the unit sphere moves on a circle parallel to the circle P_1P_3. The values of σ and τ for different positions of N moving along this circle can be obtained again from (Eq. A1.2) as

$$(\sigma - \sigma_3)(\sigma - \sigma_1) + \tau^2 = n_y^2\ (\sigma_2 - \sigma_3)\ (\sigma_2 - \sigma_1)$$

or $$\left(\sigma - \frac{\sigma_1 + \sigma_3}{2}\right)^2 + \tau^2 = n_y^2\ (\sigma_2 - \sigma_3)(\sigma_2 - \sigma_1) + \frac{(\sigma_1 - \sigma_3)^2}{4} = R_2^2$$

This describes a circle in the $\sigma - \tau$ plane with the centre at $\frac{(\sigma_1 + \sigma_3)}{2}$ and radius equal to R_2. For different values of n_y, we get a family of circles, all with centres at $\frac{(\sigma_1 + \sigma_3)}{2}$. With $n_y = 0$, we get the outermost circle. Similarly, with $n_x = \cos\alpha$ kept constant, we get another circle with centre at $\frac{(\sigma_2 + \sigma_3)}{2}$ and radius R_1. In order to determine the normal stress σ and shear stress τ on a plane with normal $\boldsymbol{n} = (n_x, n_y, n_z)$, we describe two circles with centres and radii as

centre at $\frac{\sigma_1 + \sigma_3}{2}$ and radius equal to R_2

centre at $\frac{\sigma_1 + \sigma_2}{2}$ and radius equal to R_3

where R_2 and R_3 are as given in the above equation. The intersection point of these two circles locates (σ, τ). The third circle with centre at $\frac{\sigma_2 + \sigma_3}{2}$ and radius R_1 is not an independent circle since among the three direction cosines n_x, n_y and n_z, only two are independent.

Appendix 2

The State of Pure Shear

Theorem: A necessary and sufficient condition for $\overset{n}{\boldsymbol{T}}$ to be a state of pure shear is that the first invariant should be equal to zero, i.e. $l_1 = 0$.

Proof: By definition, $\overset{n}{\boldsymbol{T}}$ is a state of pure shear at P, if there exists at least one frame of reference $Pxyz$, such that with respect to that frame,

$$\left[\tau_{ij}\right] = \begin{bmatrix} 0 & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & 0 & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & 0 \end{bmatrix}$$

Therefore, if the state of stress $\overset{n}{\boldsymbol{T}}$ is a pure shear state, then l_1, an invariant, must be equal to zero. This is therefore a necessary condition. To prove that $l_1 = 0$ is also a sufficient condition, we proceed as follows:

Given $l_1 = \sigma_x + \sigma_y + \sigma_z = 0$. Let $Px'y'z'$ be the principal axes at P. If σ_1, σ_2 and σ_3, are the principal stresses then

$$l_1 = \sigma_1 + \sigma_2 + \sigma_3 = 0 \tag{A2.1}$$

From Cauchy's formula, the normal stress σ_n on a plane $\boldsymbol{n}$ with direction cosines $n_{x'}, n_{y'}\, n_{z'}$ is

$$\sigma_n = \sigma_1 n_{x'}^2 + \sigma_2 n_{y'}^2 + \sigma_3 n_{z'}^2 \tag{A2.2}$$

We have to show that there exist at least three mutually perpendicular planes on which the normal stresses are zero. Let $\boldsymbol{n}$ be the normal to one such plane. Let $Q(x', y', z')$ be a point on this normal (Fig. A2.1).

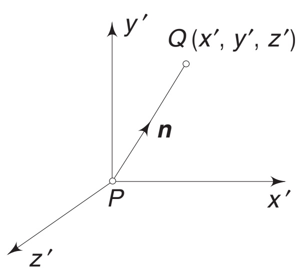

Fig. A2.1 *Normal* $\boldsymbol{n}$ *to a plane through P*

If $PQ = R$, then,

$$n_{x'} = \frac{x'}{R}, \qquad n_{y'} = \frac{y'}{R}, \qquad n_{z'} = \frac{z'}{R}$$

Since PQ is a pure shear normal, from Eq. (A2.2)

$$\sigma_1 x'^2 + \sigma_2 y'^2 + \sigma_3 z'^2 = R^2 \sigma_n = 0 \tag{A2.3}$$

The problem is to find the locus of Q. Since $l_1 = 0$, two cases are possible.

Case (i) If two of the principal stresses (say σ_1 and σ_2) are positive, the third principal stress σ_3 is negative, i.e.

$$\sigma_1 > 0, \quad \sigma_2 > 0, \quad \sigma_3 = -(\sigma_1 + \sigma_2) < 0$$

The case that σ_1 and σ_2 are negative and σ_3 is positive is similar to the above case, as the result will show.

Case (ii) One of the principal stresses (say σ_3) is zero, so that one of the remaining principal stress σ_1 is positive, and the other is negative, i.e.

$$\sigma_1 > 0, \qquad \sigma_2 = -\sigma_1 < 0, \qquad \sigma_3 = 0$$

The above two cases cover all posibilities. Let us consider case (ii) first since it is the easier one.

Case (ii) From Eq. (A2.3)

$$\sigma_1 x'^2 - \sigma_1 y'^2 = 0$$

or

$$x'^2 - y'^2 = 0$$

The solutions are

(i) $x' = 0$ and $y' = 0$. This represents the z' axis, i.e. the point Q, lies on the z' axis.

(ii) $x' = +y'$ or $x' = -y'$. These represent two mutually perpendicular planes, as shown in Fig. A2.2(a), i.e. the point Q can lie in either of these two planes.

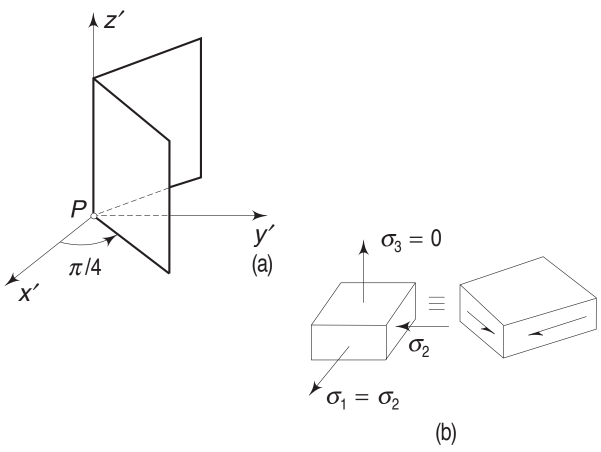

Fig. A2.2 *(a) Planes at 45° (b) Principal stress on an element under plane state of stress*

The above solutions show that for case (ii), i.e when $\sigma_3 = 0$ and $\sigma_1 = -\sigma_2$, there are three pure shear normals. These are the z' axis, an axis lying in the plane $x' = y'$ and another lying the plane $x' = -y'$. This is the elementary case usually discussed in a plane state of stress, as shown in Fig. A2.2(b).

Case (i) Since $\quad \sigma_3 = -(\sigma_1 + \sigma_2)$, Eq. (A2.3) gives

$$\sigma_1 x'^2 + \sigma_2 y'^2 - (\sigma_1 + \sigma_3) z'^2 = 0 \tag{A2.4}$$

This is the equation of an elliptic cone with vertex at P and axis along PZ' (Fig. A2.3). The point $Q(x', y', z')$ can be anywhere on the surface of the cone.

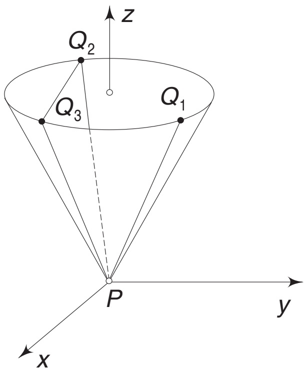

Fig. A2.3 *Cone with vertex at P and axis along PZ′*

Now one has to show that there are at least three mutually perpendicular generators of the above cone. Let $Q_1\,(x_1', y_1', 1)$ be a point on the cone and let S be a plane passing through P and perpendicular to PQ_1. We have to show that the plane S intersects the cone along PQ_2 and PQ_3 and that these two are perpendicular to each other.

Let $Q(x', y', 1)$ be a point in S. Then, S being perpendicular to PQ_1, PQ is perpendicular to PQ_1, i.e.

$$x_1'x' + y_1'y' + 1 = 0 \qquad \text{(A2.5)}$$

If Q lies on the elliptic cone also, it must satisfy Eq. (A2.4), i.e.

$$\sigma_1 x'^2 + \sigma_2 y'^2 - (\sigma_1 + \sigma_2) = 0 \qquad \text{(A 2.6)}$$

Multiply Eq. (A2.6) by $2y_1'^2$ and substitute for $y'y_1'$ from Eq. (A2.5). This gives

$$\sigma_1 x'^2\, y_1'^2 + \sigma_2 (x_1'\, x' + 1)^2 - (\sigma_1 + \sigma_2)\, y_1'^2 = 0$$

or
$$(\sigma_1\, y_1'^2 + \sigma_2\, x_1'^2)\, x'^2 + 2\sigma_2 x_1'\, x' + \left[\sigma_2 - (\sigma_1 + \sigma_2)\, y_1'^2\right] = 0 \qquad \text{(A 2.7)}$$

Similarly, multiplying Eq. (A2.6) by $x_1'^2$ and substituting for $x'x'_1$ from Eq. (A2.5), we get

$$(\sigma_2\, x_1'^2 + \sigma_1\, y_1'^2) y'^2 + 2\sigma_1\, y' y_1' + \left[\sigma_1 - (\sigma_2 + \sigma_1)\, x_1'^2\right] = 0 \qquad \text{(A2.8)}$$

If $Q(x', y_1', 1)$ is a point lying in S as well as on the cone, then it must satisfy Eqs (A2.5) and (A2.6) or equivalently Eqs (A2.7) and (A2.8). One can solve Eq. (A2.7) for x' and Eq. (A2.8) for y'. Since these are quadratic, we get two solutions for each. Let (x_2', y_2') and (x_3', y_3') be the solutions. Clearly

$$x_2'\, x_3' = \frac{\left[\sigma_2 - (\sigma_1 + \sigma_2)\, y_1'^2\right]}{\left[\sigma_2 x_1'^2 + \sigma_1 y_1'^2\right]} \qquad \text{(A 2.9)}$$

$$y_2'\, y_3' = \frac{\left[\sigma_1 - (\sigma_2 + \sigma_1)\, x_1'^2\right]}{\left[\sigma_2 x_1'^2 + \sigma_1 y_1'^2\right]} \qquad \text{(A 2.10)}$$

Adding the above two equations

$$x_2'\, x_3' + y_2'\, y_3' = \frac{\sigma_1 + \sigma_2 - \sigma_1 y_1'^2 - \sigma_2 y_1'^2 - \sigma_2 x_1'^2 - \sigma_1 x_1'^2}{\sigma_2 x_1'^2 + \sigma_1 y_1'^2}$$

Since $Q_1(x_1', y_1', 1)$ is on the cone and recalling that $\sigma_1 + \sigma_2 = -\sigma_3$, the right-hand side is equal to –1, i.e.

$$x_2' \, x_3' + y_2' \, y_3' + 1 = 0$$

Consequently, PQ_2 and PQ_3 are perpendicular to each other if $Q_2 = (x_2', \; y_2', \; 1)$ and $Q_3 \; (x_3', y_3', 1)$ are real. If x_2', x_3' and y_2', y_3', the solutions of Eqs (A2.7) and (A2.8), are to be real, then the descriminants must be greater than zero. For this, let $Q_1 (x_1', y_1', 1)$ be specifically $Q_1(1, 1, 1)$ i.e. choose $x_1' = y_1' = 1$. Both the descriminants of Eqs (A2.7) and (A2.8) then are

$$4(\sigma_1^2 + \sigma_1\sigma_2 + \sigma_2^2)$$

The above quantity is greater than zero, since $\sigma_1 > 0$ and $\sigma_2 > 0$. Therefore, x_2', x_3' and y_2', y_3' are real.

Appendix 3

Stress Quadric of Cauchy

Let $\overset{n}{\boldsymbol{T}}$ be the resultant stress vector at point P (see Fig. A3.1) on a plane with unit normal $\boldsymbol{n}$. The stress surface S associated with a given state of stress $\overset{n}{\boldsymbol{T}}$ is defined as the locus of all points Q, such that

$$\boldsymbol{PQ} = R\boldsymbol{n}$$

where
$$R = |\boldsymbol{PQ}| = \frac{1}{\left(|\sigma(\boldsymbol{n})|\right)^{1/2}}$$

and $\sigma(\boldsymbol{n})$ is the normal stress component on the plane $\boldsymbol{n}$. This means that a point Q is chosen along $\boldsymbol{n}$ such that $R = 1/\sqrt{\sigma}$. If such Qs are marked for every plane passing through P, then we get a surface S and this surface determines the normal component of stress on any plane through P. The surface consists of S_t and S_c—the tensile and the compressive branches of the surface.

The normal to the surface S at $Q(n)$ is parallel to $\overset{n}{\boldsymbol{T}}$. Thus, S completely determines the state of stress at P. The following cases are possible.

Case (i) $\sigma_1 \neq 0$, $\sigma_2 \neq 0$, $\sigma_3 \neq 0$; S_t and S_c are each a central quadric surface about $\boldsymbol{P}$ with axes along $\boldsymbol{n}_x$, $\boldsymbol{n}_y$ and $\boldsymbol{n}_z$.

(i) If σ_1, σ_2 and σ_3 all have the same sign, say $\sigma_1 > 0$, $\sigma_2 > 0$, $\sigma_3 > 0$ then $S = S_t$ is an ellipsoid with axes along $\boldsymbol{n}_x$, $\boldsymbol{n}_y$ and $\boldsymbol{n}_x$ at $\boldsymbol{P}$. There are two cases
  (a) If $\sigma_1 = \sigma_2 \neq \sigma_3$, then $S = S_t$ is a spheroid with polar axis along $\boldsymbol{n}_z$
  (b) If $\sigma_1 = \sigma_2 = \sigma_3$, then $S = S_t$ is a sphere.

(ii) If σ_1, σ_2 and σ_3 are not all of the same sign, say $\sigma_1 > 0$, $\sigma_2 > 0$ and $\sigma_3 < 0$, then S_t is a hyperboloid with one sheet and S_c is a double sheeted hyperboloid, the vertices of which are along the n_z axis. In particular, if $\sigma_1 = \sigma_2$, then S_t and S_c are hyperboloids of revolution with a polar axis along n_z.

Case (ii) Let $\sigma_1 \neq 0$, $\sigma_2 \neq 0$ and $\sigma_3 = 0$ (i.e. plane state). The S_t and S are right second-order cylinders whose generators are parallel to n_z and whose cross-sections have axes along $\boldsymbol{n}_x$ and $\boldsymbol{n}_y$. In this case, two possibilities can be considered.

(i) If $\sigma_1 > 0$, $\sigma_2 > 0$, then $S = S_t$ is an elliptic cylinder. In particular, If $\sigma_1 = \sigma_2$ then $S = S_t$ is a circular cylinder.

(ii) If $\sigma_1 > 0$ and $\sigma_2 < 0$, then S_t is a hyperbolic cylinder whose cross-section has vertices on the $\boldsymbol{n}_x$ axis and S_c is a hyperbolic cylinder.

Case (iii) If $\sigma_1 \neq 0$ and $\sigma_2 = \sigma_3 = 0$ (uniaxial state) and say $\sigma_1 > 0$ then $S = S_t$ consists of two parallel planes, each perpendicular to $\boldsymbol{n}_x$ and equidistant from P.

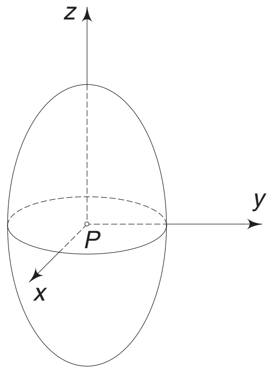

Fig. A3.1 *Ellipsoidal surface*

One can prove the above statements directly from Eqs (1.53) of Sec. 1.23. These equations are

$$S_t : \sigma_1 x^2 + \sigma_2 y^2 + \sigma_3 z^3 = 1$$
$$S_c : \sigma_1 x^2 + \sigma_2 y^2 + \sigma_3 z^2 = -1$$

These can be rewritten as

$$S_t : \frac{x^2}{(1/\sqrt{\sigma_1})^2} + \frac{y^2}{(1/\sqrt{\sigma_2})^2} + \frac{z^2}{(1/\sqrt{\sigma_3})^2} = 1$$

$$S_c : \frac{x^2}{(1/\sqrt{\sigma_1})^2} + \frac{y^2}{(1/\sqrt{\sigma_2})^2} + \frac{z^2}{(1/\sqrt{\sigma_3})^2} = -1 \quad \text{(A3.1)}$$

Case (i) $\sigma_1 \neq 0$, $\sigma_2 \neq 0$, $\sigma_3 \neq 0$

(i) $\sigma_1 > 0$, $\sigma_2 > 0$, $\sigma_3 > 0$

Equation (A3.1) shows that S_c is an imaginary surface and hence, $S = S_t$. This equation represents an ellipsoid.

(a) If $\sigma_1 = \sigma_2 \neq \sigma_3$ the central section is a circle

(b) If $\sigma_1 = \sigma_2 = \sigma_3$ the surface is a sphere

(ii) If $\sigma_1 > 0$, $\sigma_2 > 0$, $\sigma_3 < 0$

$$S_t : \frac{x^2}{(1/\sqrt{\sigma_1})^2} + \frac{y^2}{(1/\sqrt{\sigma_2})^2} - \frac{z^2}{(1/\sqrt{\sigma_3})^2} = 1$$

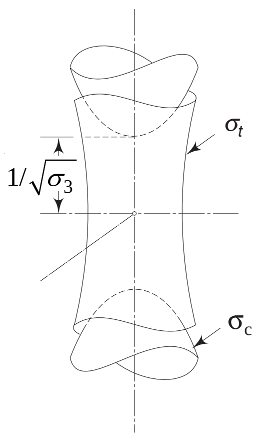

Fig. A3.2 *One-sheeted and two sheeted hyperboloids*

$$S_c : -\frac{x^2}{(1/\sqrt{\sigma_1})^2} - \frac{y^2}{(1/\sqrt{\sigma_2})^2} + \frac{z^2}{(1/\sqrt{\sigma_3})^2} = 1 \quad \text{(A3.2)}$$

Hence, S_t is a one-sheeted hyperboloid and S_c is a two-sheeted hyperbloid. This is shown in Fig. A3.2.

Case (ii) Let $\sigma_1 \neq 0$, $\sigma_2 \neq 0$ and $\sigma_3 = 0$. Then Eq. (1.53) reduces to

$$\sigma_1 x^2 + \sigma_2 y^2 = \pm 1 \quad \text{(A3.3)}$$

This is obviously a second-order cylinder, the surface of which is made of straight lines parallel to the *z*-axis, passing through every point of the curve in the *xy* plane, of which an equation in that plane is expressed by Eq. (A3.3).

(i) If $\sigma_1 > 0$ and $\sigma_2 > 0$, the above equation becomes

$$\sigma_1 x^2 + \sigma_2 y^2 + 1$$

or

$$\frac{x^2}{(1/\sigma_1)^2} + \frac{y^2}{(1/\sigma_2)^2} = 1$$

This is the equation of an ellipse in xy plane. Hence, $S = S_t$ is an elliptic cylinder.

In particular, if $\sigma_1 = \sigma_2$, the elliptic cylinder becomes a circular cylinder.

(ii) If $\sigma_1 > 0$ and $\sigma_2 < 0$, then the equation becomes

$$\sigma_1 x^2 - |\sigma_2|\, y^2 = \pm 1$$

or

$$x^2/(1/\sigma_1)^2 - y^2\,(1/\sigma_2^2) = \pm 1$$

This describes conjugate hyperbolas in the xy plane. S_t is given by a hyperbolic cylinder, the cross-sectional vertices of which lie on the $\boldsymbol{n}_x$ axis and S_c is given by a hyperbolic cylinder with its cross-sectional vertices lying on the $\boldsymbol{n}$ axis.

Case (iii) If $\sigma_1 \neq 0$, $\sigma_2 = \sigma_3 = 0$, Eq. (1.53) reduces to

$$\sigma_1\, x^2 = \pm 1$$

When $\sigma_1 > 0$, this becomes

$$x^2 = 1/\sigma_1$$

or

$$x = \pm 1/\sqrt{\sigma_1}$$

This represents two straight lines parallel to the y axis and equidistant from it. Hence, $S = S_t$ is given by two parallel planes, each perpendicular to $\boldsymbol{n}_x$ and equidistant from P.

CHAPTER 2

Analysis of Strain

2.1 INTRODUCTION

In this chapter the state of strain at a point will be analysed. In elementary strength of materials two types of strains were introduced: (i) the extensional strain (in x or y direction) and (ii) the shear strain in the xy plane. Figure 2.1 illustrates these two simple cases of strain. In each case, the initial or undeformed position of the element is indicated by full lines and the changed position by dotted lines. These are two-dimensional strains.

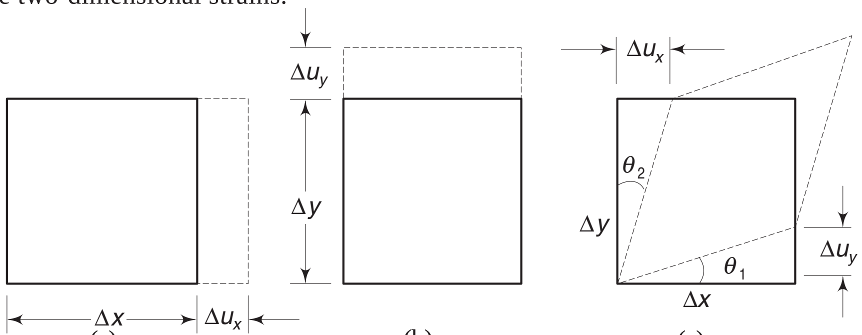

Fig. 2.1 *(a) Linear strain in x direction (b) linear strain in y direction (c) shear strain in xy plane*

In Fig. 2.1(a), the element undergoes an extension Δu_x in x direction. The extensional or linear strain is defined as the change in length per unit initial length. If ε_x denotes the linear strain in x direction, then

$$\varepsilon_x = \frac{\Delta u_x}{\Delta x} \tag{2.1}$$

Similarly, the linear strain in y direction [Fig. 2.1(b)] is

$$\varepsilon_y = \frac{\Delta u_y}{\Delta y} \tag{2.2}$$

Figure 2.1(c) shows the shear strain γ_{xy} in the xy plane. Shear strain γ_{xy} is defined as the change in the initial right angle between two line elements originally parallel to the x and y axes. In the figure, the total change in the angle is $\theta_1 + \theta_2$. If θ_1 and θ_2 are very small, then one can put

$$\theta_1 \text{ (in radians)} + \theta_2 \text{ (in radians)} = \tan \theta_1 + \tan \theta_2$$

From Fig. 2.1(c)

$$\tan \theta_1 = \frac{\Delta u_y}{\Delta x}, \qquad \tan \theta_2 = \frac{\Delta u_x}{\Delta y} \tag{2.3}$$

Therefore, the shear strain γ_{xy} is

$$\gamma_{xy} = \theta_1 + \theta_2 = \frac{\Delta u_y}{\Delta x} + \frac{\Delta u_x}{\Delta y} \tag{2.4}$$

Reduction in the initial right angle is considered to be a positive shear strain, since positive shear stress components τ_{xy} and τ_{yx} cause a decrease in the right angle.

In addition to these two types of strains, a third type of strain, called the volumetric strain, was also introduced in elementary strength of materials. This is change in volume per unit original volume. In this chapter, we will study strains in three dimensions and we will begin with the study of deformations.

2.2 DEFORMATIONS

In order to study deformation or change in the shape of a body, we compare the positions of material points before and after deformation. Let a point P belonging to the body and having coordinates (x, y, z) be displaced after deformations to P' with coordinates (x', y', z') (Fig. 2.2). Since P is displaced to P', the vector segment $\boldsymbol{PP'}$ is called the displacement vector and is denoted by $\boldsymbol{u}$.

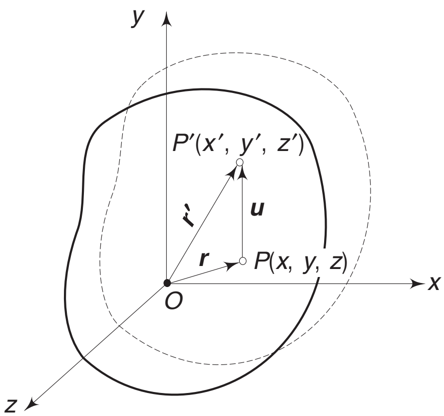

Fig. 2.2 *Displacement of point P to P′*

The displacement vector $\boldsymbol{u}$ has components u_x, u_y and u_z along the x, y and z axes respectively, and one can write

$$\boldsymbol{u} = \boldsymbol{i}u_x + \boldsymbol{j}u_y + \boldsymbol{k}u_z \tag{2.5}$$

The displacement undergone by any point is a function of its initial co-ordinates. We assume that the displacement is defined throughout the volume of the body, i.e. the displacement vector $\boldsymbol{u}$ (both in magnitude and direction) of any point P belonging to the body is known once its coordinates are known. Then we can say that a displacement vector field has been defined throughout the volume of the body. If $\boldsymbol{r}$ is the position vector of point P, and $\boldsymbol{r}'$ that of point P', then

$$\begin{aligned} \boldsymbol{r}' &= \boldsymbol{r} + \boldsymbol{u} \\ \boldsymbol{u} &= \boldsymbol{r}' - \boldsymbol{r} \end{aligned} \tag{2.6}$$

Example 2.1 *The displacement field for a body is given by*

$$\boldsymbol{u} = (x^2 + y)\boldsymbol{i} + (3 + z)\boldsymbol{j} + (x^2 + 2y)\boldsymbol{k}$$

What is the deformed position of a point originally at (3, 1, –2)?

Solution The displacement vector $\boldsymbol{u}$ at (3, 1, –2) is

$$\begin{aligned}\boldsymbol{u} &= (3^2 + 1)\boldsymbol{i} + (3 - 2)\boldsymbol{j} + (3^2 + 2)\boldsymbol{k}\\ &= 10\boldsymbol{i} + \boldsymbol{j} + 11\boldsymbol{k}\end{aligned}$$

The initial position vector $\boldsymbol{r}$ of point P is

$$\boldsymbol{r} = 3\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$$

The final position vector $\boldsymbol{r}'$ of point P' is

$$\boldsymbol{r}' = \boldsymbol{r} + \boldsymbol{u} = 13\boldsymbol{i} + 2\boldsymbol{j} + 9\boldsymbol{k}$$

Example 2. 2 *Two points P and Q in the undeformed body have coordinates (0, 0, 1) and (2, 0, –1) respectively. Assuming that the displacement field given in Example 2.1 has been imposed on the body, what is the distance between points P and Q after deformation?*

Solution The displacement vector at point P is

$$\boldsymbol{u}(P) = (0 + 0)\boldsymbol{i} + (3 + 1)\boldsymbol{j} + (0 + 0)\boldsymbol{k} = 4\boldsymbol{j}$$

The displacement components at P are $u_x = 0$, $u_y = 4$, $u_z = 0$. Hence, the final coordinates of P after deformation are

$$\begin{aligned}P' : x + u_x &= 0 + 0 = 0\\ y + u_y &= 0 + 4 = 4\\ z + u_z &= 1 + 0 = 1\end{aligned}$$

or $$P': (0, 4, 1)$$

Similarly, the displacement components at point Q are,

$$\boldsymbol{u}_x = 4, \quad \boldsymbol{u}_y = 2, \quad \boldsymbol{u}_z = 4$$

and the coordinates of Q' are (6, 2, 3).
The distance $P'Q'$ is therefore

$$d' = (6^2 + 2^2 + 2^2)^{1/2} = 2\sqrt{11}$$

2.3 DEFORMATION IN THE NEIGHBOURHOOD OF A POINT

Let P be a point in the body with coordinates (x, y, z). Consider a small region surrounding the point P. Let Q be a point in this region with coordinates $(x + \Delta x, y + \Delta y, z + \Delta z)$. When the body undergoes deformation, the points P and Q move to P' and Q'. Let the displacement vector $\boldsymbol{u}$ at P have components (u_x, u_y, u_z) (Fig. 2.3).

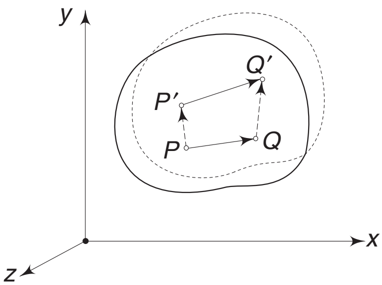

Fig. 2.3 *Displacements of two neighbouring points P and Q*

The coordinates of P, P' and Q are

P: (x, y, z)

P': $(x + u_x, y + u_y, z + u_z)$

Q: $(x + \Delta x, y + \Delta y, z + \Delta z)$

The displacement components at Q differ slightly from those at P since Q is away from P by Δx, Δy and Δz. Consequently, the displacements at Q are,

$$u_x + \Delta u_x,\ u_y + \Delta u_y,\ u_z + \Delta u_z.$$

If Q is very close to P, then to first-order approximation

$$\Delta u_x = \frac{\partial u_x}{\partial x}\Delta x + \frac{\partial u_x}{\partial y}\Delta y + \frac{\partial u_x}{\partial z}\Delta z \tag{2.7a}$$

The first term on the right-hand side is the rate of increase of u_x in x direction multiplied by the distance traversed, Δx. The second term is the rate of increase of u_x in y direction multiplied by the distance traversed in y direction, i.e. Δy. Similarly, we can also interpret the third term. For Δu_y and Δu_z too, we have

$$\Delta u_y = \frac{\partial u_y}{\partial x}\Delta x + \frac{\partial u_y}{\partial y}\Delta y + \frac{\partial u_y}{\partial z}\Delta z \tag{2.7b}$$

$$\Delta u_z = \frac{\partial u_z}{\partial x}\Delta x + \frac{\partial u_z}{\partial y}\Delta y + \frac{\partial u_z}{\partial z}\Delta z \tag{2.7c}$$

Therefore, the coordinates of Q' are,

$$Q' = (x + \Delta x + u_x + \Delta u_x,\ y + \Delta y + u_y + \Delta u_y,\ z + \Delta z + u_z + \Delta u_z) \tag{2.8}$$

Before deformation, the segment PQ had components Δx, Δy and Δz along the three axes. After deformation, the segment $P'Q'$ has components $\Delta x + \Delta u_x$, $\Delta y + \Delta u_y$, $\Delta z + \Delta u_z$ along the three axes. Terms like,

$$\frac{\partial u_x}{\partial x}, \frac{\partial u_x}{\partial y}, \frac{\partial u_x}{\partial z}, \text{ etc.}$$

are important in the analysis of strain. These are the gradients of the displacement components (at a point P) in x, y and z directions. One can represent these in the form of a matrix called the displacement-gradient matrix as

$$\left[\frac{\partial u_i}{\partial x_j}\right] = \begin{bmatrix} \dfrac{\partial u_x}{\partial x} & \dfrac{\partial u_x}{\partial y} & \dfrac{\partial u_x}{\partial z} \\ \dfrac{\partial u_y}{\partial x} & \dfrac{\partial u_y}{\partial y} & \dfrac{\partial u_y}{\partial z} \\ \dfrac{\partial u_z}{\partial x} & \dfrac{\partial u_z}{\partial y} & \dfrac{\partial u_z}{\partial z} \end{bmatrix}$$

Example 2.3 *The following displacement field is imposed on a body*

$$\boldsymbol{u} = (xy\boldsymbol{i} + 3x^2z\boldsymbol{j} + 4\boldsymbol{k})10^{-2}$$

Consider a point P and a neighbouring point Q where PQ has the following direction cosines

$$n_x = 0.200, \quad n_y = 0.800, \quad n_z = 0.555$$

Point P has coordinates (2, 1, 3). If PQ = Δs, find the components of ***P′Q′*** *after deformation.*

Solution Before deformation, the components of PQ are

$$\Delta x = n_x \, \Delta s = 0.2 \, \Delta s$$
$$\Delta y = n_y \, \Delta s = 0.8 \, \Delta s$$
$$\Delta z = n_z \, \Delta s = 0.555 \, \Delta s$$

Using Eqs (2.7a)–(2.7c), the values of Δu_x, Δu_y and Δu_z can be calculated. We are using $p = 10^{-2}$;

$$u_x = pxy \qquad u_y = 3px^2z \qquad u_z = 4p$$

$$\frac{\partial u_x}{\partial x} = py \qquad \frac{\partial u_y}{\partial x} = 6\,pxz \qquad \frac{\partial u_z}{\partial x} = 0$$

$$\frac{\partial u_x}{\partial y} = px \qquad \frac{\partial u_y}{\partial y} = 0 \qquad \frac{\partial u_z}{\partial y} = 0$$

$$\frac{\partial u_x}{\partial z} = 0 \qquad \frac{\partial u_y}{\partial z} = 3px^2 \qquad \frac{\partial u_z}{\partial z} = 0$$

At point $P(2, 1, 3)$ therefore,

$$\Delta u_x = (y\Delta x + x\Delta y)p = (\Delta x + 2\Delta y)p$$
$$\Delta u_y = (6xz\Delta x + 3x^2\Delta z)p = (36\Delta x + 12\Delta z)p$$
$$\Delta u_z = 0$$

Substituting for Δx, Δy and Δz, the components of $\Delta s' = |\boldsymbol{P'Q'}|$ are

$$\Delta x + \Delta u_x = 1.01\,\Delta x + 0.02\,\Delta y = (0.202 + 0.016)\,\Delta s = 0.218\,\Delta s$$
$$\Delta y + \Delta u_y = (0.36\,\Delta x + \Delta y + 0.12\,\Delta z) = (0.072 + 0.8 + 0.067)\,\Delta s$$
$$= 0.939\,\Delta s$$
$$\Delta z + \Delta u_z = \Delta z = 0.555\,\Delta s$$

Hence, the new vector $\boldsymbol{P'Q'}$ can be written as

$$\boldsymbol{P'Q'} = (0.218\boldsymbol{i} + 0.939\boldsymbol{j} + 0.555\boldsymbol{k})\Delta s$$

2.4 CHANGE IN LENGTH OF A LINEAR ELEMENT

Deformation causes a point $P(x, y, z)$ in the solid body under consideration to be displaced to a new position P' with coordinates $(x + u_x, y + u_y, z + u_z)$ where u_x, u_y and u_z are the displacement components. A neighbouring point Q with coordinates $(x + \Delta x, y + \Delta y, z + \Delta z)$ gets displaced to Q' with new coordinates $(x + \Delta x + u_x + \Delta u_x, y + \Delta y + u_y + \Delta u_y, z + \Delta z + u_z + \Delta u_z)$. Hence, it is possible to determine the change in the length of the line element PQ caused by deformation. Let Δs be the length of the line element PQ. Its components are

$$\Delta s: (\Delta x, \Delta y, \Delta z)$$

$$\therefore \qquad \Delta s^2: (PQ)^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$$

Let $\Delta s'$ be the length of $P'Q'$. Its components are

$$\Delta s': (\Delta x' = \Delta x + \Delta u_x, \Delta y' = \Delta y + \Delta u_y, \Delta z' = \Delta z + \Delta u_z)$$

$$\therefore \qquad \Delta s'^2: (P'Q')^2 = (\Delta x + \Delta u_x)^2 + (\Delta y + \Delta u_y)^2 + (\Delta z + \Delta u_z)^2$$

From Eqs (2.7a)–(2.7c),

$$\begin{aligned}
\Delta x' &= \left(1 + \frac{\partial u_x}{\partial x}\right)\Delta x + \frac{\partial u_x}{\partial y}\Delta y + \frac{\partial u_x}{\partial z}\Delta z \\
\Delta y' &= \frac{\partial u_y}{\partial x}\Delta x + \left(1 + \frac{\partial u_y}{\partial y}\right)\Delta y + \frac{\partial u_y}{\partial z}\Delta z \\
\Delta z' &= \frac{\partial u_z}{\partial x}\Delta x + \frac{\partial u_z}{\partial y}\Delta y + \left(1 + \frac{\partial u_z}{\partial z}\right)\Delta z
\end{aligned} \tag{2.9}$$

We take the difference between $\Delta s'^2$ and Δs^2

$$\begin{aligned}
(P'Q')^2 - (PQ)^2 &= \Delta s'^2 - \Delta s^2 \\
&= (\Delta x'^2 + \Delta y'^2 + \Delta z'^2) - (\Delta x^2 + \Delta y^2 + \Delta z^2) \\
&= 2(E_{xx}\Delta x^2 + E_{yy}\Delta y^2 + E_{zz}\Delta z^2 + E_{xy}\Delta x\,\Delta y \\
&\qquad + E_{yz}\Delta y\,\Delta z + E_{xz}\Delta x\,\Delta z)
\end{aligned} \tag{2.10}$$

where

$$\begin{aligned}
E_{xx} &= \frac{\partial u_x}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u_x}{\partial x}\right)^2 + \left(\frac{\partial u_y}{\partial x}\right)^2 + \left(\frac{\partial u_z}{\partial x}\right)^2\right] \\
E_{yy} &= \frac{\partial u_y}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial u_x}{\partial y}\right)^2 + \left(\frac{\partial u_y}{\partial y}\right)^2 + \left(\frac{\partial u_z}{\partial y}\right)^2\right] \\
E_{zz} &= \frac{\partial u_z}{\partial z} + \frac{1}{2}\left[\left(\frac{\partial u_x}{\partial z}\right)^2 + \left(\frac{\partial u_y}{\partial z}\right)^2 + \left(\frac{\partial u_z}{\partial z}\right)^2\right] \\
E_{xy} &= \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial x}\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial x}\frac{\partial u_z}{\partial y} \\
E_{yz} &= \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} + \frac{\partial u_x}{\partial y}\frac{\partial u_x}{\partial z} + \frac{\partial u_y}{\partial y}\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\frac{\partial u_z}{\partial z} \\
E_{xz} &= \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial x}\frac{\partial u_x}{\partial z} + \frac{\partial u_y}{\partial x}\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial x}\frac{\partial u_z}{\partial z}
\end{aligned} \tag{2.11}$$

It is observed that

$$E_{xy} = E_{yx}, \qquad E_{yz} = E_{zy}, \qquad E_{xz} = E_{zx}$$

We introduce the notation

$$E_{PQ} = \frac{\Delta s' - \Delta s}{\Delta s} \tag{2.12}$$

E_{PQ} is the ratio of the increase in distance between the points P and Q caused by the deformation to their initial distance. This quantity will be called the relative extension at point P in the direction of point Q. Now,

$$\frac{\Delta s'^2 - \Delta s^2}{2} = \left(\frac{\Delta s' - \Delta s}{\Delta s} + \frac{(\Delta s' - \Delta s)^2}{2\Delta s^2}\right)\Delta s^2$$

$$= \left(E_{PQ} + \frac{1}{2}E_{PQ}^2\right)\Delta s^2 \tag{2.13}$$

$$= E_{PQ}\left(1 + \frac{1}{2}E_{PQ}\right)\Delta s^2$$

From Eq. (2.10), substituting for $(\Delta s'^2 - \Delta s^2)$

$$E_{PQ}\left(1 + \frac{1}{2}E_{PQ}\right)\Delta s^2 = E_{xx}\,\Delta x^2 + E_{yy}\,\Delta y^2 + E_{zz}\,\Delta z^2$$
$$+ E_{xy}\,\Delta x\,\Delta y + E_{yz}\,\Delta y\,\Delta z + E_{xz}\,\Delta x\,\Delta z$$

If n_x, n_y and n_z are the direction cosines of PQ, then

$$n_x = \frac{\Delta x}{\Delta s}, \qquad n_y = \frac{\Delta y}{\Delta s}, \qquad n_z = \frac{\Delta z}{\Delta s}$$

Substituting these in the above expression

$$E_{PQ}\left(1 + \frac{1}{2}E_{PQ}\right) = E_{xx}n_x^2 + E_{yy}n_y^2 + E_{zz}n_z^2 + E_{xy}n_x n_y$$
$$+ E_{yz}n_y n_z + E_{xz}n_x n_z \tag{2.14}$$

Equation (2.14) gives the value of the relative extension at point P in the direction PQ with direction cosines n_x, n_y and n_z .

If the line segment PQ is parallel to the x axis before deformation, then $n_x = 1$, $n_y = n_z = 0$ and

$$E_x\left(1 + \frac{1}{2}E_x\right) = E_{xx} \tag{2.15}$$

Hence,
$$E_x = [1 + 2E_{xx}]^{1/2} - 1 \tag{2.16}$$

This gives the relative extension of a line segment originally parallel to the x-axis. By analogy, we get

$$E_y = [1 + 2E_{yy}]^{1/2} - 1, \qquad E_z = [1 + 2E_{zz}]^{1/2} - 1 \tag{2.17}$$

2.5 CHANGE IN LENGTH OF A LINEAR ELEMENT—LINEAR COMPONENTS

Equation (2.11) in the previous section contains linear quantities like $\partial u_x/\partial x$, $\partial u_y/\partial y$, $\partial u_x/\partial y$, . . ., etc., as well as non-linear terms, like $(\partial u_x/\partial x)^2$, $(\partial u_x/\partial x \cdot \partial u_x/\partial y)$, …, etc. If the deformation imposed on the body is small, the quantities like $\partial u_x/\partial x$, $\partial u_y/\partial y$, etc. are extremely small so that their squares and products can be neglected. Retaining only linear terms, the following equations are obtained

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \qquad \varepsilon_{yy} = \frac{\partial u_y}{\partial y}, \qquad \varepsilon_{zz} = \frac{\partial u_z}{\partial z} \tag{2.18}$$

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}, \qquad \gamma_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}, \qquad \gamma_{xz} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \tag{2.19}$$

$$E_{PQ} \approx \varepsilon_{PQ} = \varepsilon_{xx} n_x^2 + \varepsilon_{yy} n_y^2 + \varepsilon_{zz} n_z^2 + \varepsilon_{xy} n_x n_y + \varepsilon_{yz} n_y n_z + \varepsilon_{xz} n_x n_z \tag{2.20}$$

Equation 2.20 directly gives the linear strain at point P in the direction PQ with direction cosines n_x, n_y, n_z. When $n_x = 1$, $n_y = n_z = 0$, the line element PQ is parallel to the x axis and the linear strain is

$$E_x \approx \varepsilon_{xx} = \frac{\partial u_x}{\partial x}$$

Similarly, $$E_y \approx \varepsilon_{yy} = \frac{\partial u_y}{\partial y} \quad \text{and} \quad E_z \approx \varepsilon_{zz} = \frac{\partial u_z}{\partial z}$$

are the linear strains in y and z directions respectively. In the subsequent analyses, we will use only the linear terms in strain components and neglect squares and products of strain components. The relations expressed by Eqs (2.18) and (2.19) are known as the strain displacement relations of Cauchy.

2.6 RECTANGULAR STRAIN COMPONENTS

ε_{xx}, ε_{yy} and ε_{zz} are the linear strains at a point in x, y and z directions. It will be shown later that γ_{xy}, γ_{yz} and γ_{xz} represent shear strains in xy, yz and xz planes respectively. Analogous to the rectangular stress components, these six strain components are called the rectangular strain components at a point.

2.7 THE STATE OF STRAIN AT A POINT

Knowing the six rectangular strain components at a point P, one can calculate the linear strain in any direction PQ, using Eq. (2.20). The totality of all linear strains in every possible direction PQ defines the state of strain at point P. This definition is similar to that of the state of stress at a point. Since all that is required to determine the state of strain are the six rectangular strain components, these six components are said to define the state of strain at a point. We can write this as

$$[\varepsilon_{ij}] = \begin{bmatrix} \varepsilon_{xx} & \gamma_{xy} & \gamma_{xz} \\ \gamma_{xy} & \varepsilon_{yy} & \gamma_{yz} \\ \gamma_{xz} & \gamma_{yz} & \varepsilon_{zz} \end{bmatrix} \tag{2.21}$$

To maintain consistency, we could have written

$$\varepsilon_{xy} = \gamma_{xy}, \qquad \varepsilon_{yz} = \gamma_{yz}, \qquad \varepsilon_{xz} = \gamma_{xz}$$

but as it is customary to represent the shear strain by γ, we have retained this notation. In the theory of elasticity, $1/2\gamma_{xy}$ is written as e_{xy}, i.e.

$$\frac{1}{2}\gamma_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = e_{xy} \tag{2.22}$$

If we follow the above notation and use

$$e_{xx} = \varepsilon_{xx}, \qquad e_{yy} = \varepsilon_{yy}, \qquad e_{zz} = \varepsilon_{zz}$$

then Eq. (2.20) can be written in a very short form as

$$\varepsilon_{PQ} = \sum_i \sum_j e_{ij}\, n_i n_j$$

where i and j are summed over x, y and z, Note that $e_{ij} = e_{ji}$

2.8 INTERPRETATION OF γ_{xy}, γ_{yz}, γ_{xz} AS SHEAR STRAIN COMPONENTS

It was shown in the previous section that

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \qquad \varepsilon_{yy} = \frac{\partial u_y}{\partial y}, \qquad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}$$

represent the linear strains of line elements parallel to the x, y and z axes respectively. It was also stated that

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}, \qquad \gamma_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}, \qquad \gamma_{xz} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}$$

represent the shear strains in the xy, yz and xz planes respectively. This can be shown as follows.

Consider two line elements, PQ and PR, originally perpendicular to each other and parallel to the x and y axes respectively (Fig. 2.4a).

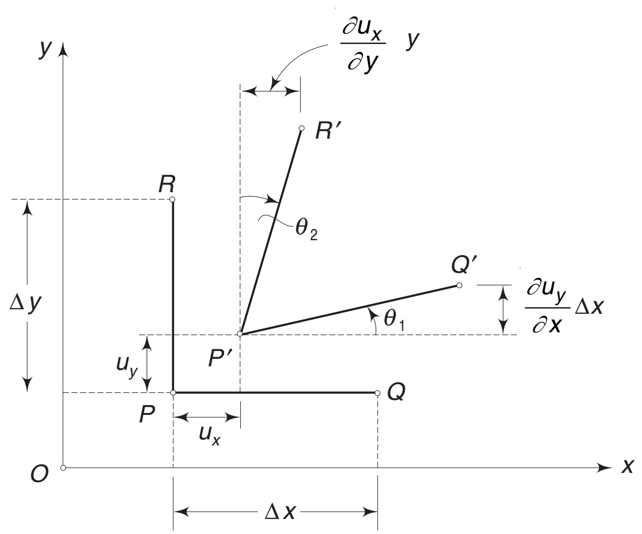

Fig. 2.4 *(a) Change in orientations of line segments PQ and PR-shear strain*

Let the coordinates of P be (x, y) before deformation and let the lengths of PQ and PR be Δx and Δy respectively. After deformation, point P moves to P', point Q to Q' and point R to R'.

Let u_x, u_y be the displacements of point P, so that the coordinates of P' are $(x + u_x,\ y + u_y)$. Since point Q is Δx distance away from P, the displacement components of $Q(x + \Delta x, y)$ are

$$u_x + \frac{\partial u_x}{\partial x}\Delta x \quad \text{and} \quad u_y + \frac{\partial u_y}{\partial x}\Delta x$$

Similarly, the displacement components of $R(x, y + \Delta y)$ are

$$u_x + \frac{\partial u_x}{\partial y}\Delta y \quad \text{and} \quad u_y + \frac{\partial u_y}{\partial y}\Delta y$$

From Fig. 2.4(a), it is seen that if θ_1 and θ_2 are small, then

$$\theta_1 \approx \tan\theta_1 = \frac{\partial u_y}{\partial x}$$

$$\theta_2 \approx \tan\theta_2 = \frac{\partial u_x}{\partial y} \tag{2.23}$$

so that the total change in the original right angle is

$$\theta_1 + \theta_2 = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = \gamma_{xy} \tag{2.24}$$

This is the shear strain in the xy plane at point P. Similarly, the shear strains γ_{yz} and γ_{zx} can be interpreted appropriately.

If the element PQR undergoes a pure rigid body rotation through a small angular displacement, then from Fig. 2.4(b) we note

$$\omega_{zo} = \frac{\partial u_y}{\partial x} = -\frac{\partial u_x}{\partial y}$$

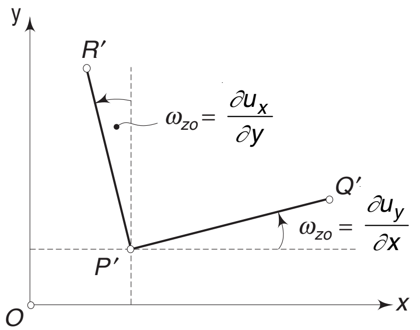

Fig. 2.4 *(b) Change in orientations of line segments PQ and PR-rigid body rotation*

taking the counter-clockwise rotation as positive. The negative sign in $(-\partial u_x/\partial y)$ comes since a positive $\partial u_x/\partial y$ will give a movement from the y to the x axis as shown in Fig. 2.4(a). No strain occurs during this rigid body displacement. We define

$$\omega_z = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) = \omega_{yx} \tag{2.25}$$

This represents the average of the sum of rotations of the x and y elements and is called the rotational component. Similarly, for rotations about the x and y axes, we get

$$\omega_x = \frac{1}{2}\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right) = \omega_{zy} \tag{2.26}$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) = \omega_{xz} \tag{2.27}$$

Example 2.4 *Consider the displacement field*

$$\boldsymbol{u} = [y^2\boldsymbol{i} + 3yz\boldsymbol{j} + (4 + 6x^2)\boldsymbol{k}]10^{-2}$$

What are the rectangular strain components at the point P(1, 0, 2)? Use only linear terms.

Solution $\quad u_x = y^2 \cdot 10^{-2} \qquad u_y = 3yz \cdot 10^{-2} \qquad u_z = (4 + 6x^2) \cdot 10^{-2}$

$$\frac{\partial u_x}{\partial x} = 0 \qquad \frac{\partial u_y}{\partial x} = 0 \qquad \frac{\partial u_z}{\partial x} = 12x \cdot 10^{-2}$$

$$\frac{\partial u_x}{\partial y} = 2y \cdot 10^{-2} \qquad \frac{\partial u_y}{\partial y} = 3z \cdot 10^{-2} \qquad \frac{\partial u_z}{\partial y} = 0$$

$$\frac{\partial u_x}{\partial z} = 0 \qquad \frac{\partial u_y}{\partial z} = 3y \cdot 10^{-2} \qquad \frac{\partial u_z}{\partial z} = 0$$

The linear strains at (1, 0, 2) are

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = 0, \qquad \varepsilon_{yy} = \frac{\partial u_y}{\partial y} = 6 \times 10^{-2}, \qquad \varepsilon_{zz} = \frac{\partial u_z}{\partial z} = 0$$

The shear strains at (1, 0, 2) are

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = 0 + 0 = 0$$

$$\gamma_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} = 0 + 0 = 0$$

$$\gamma_{xz} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} = 0 + 12 \times 10^{-2} = 12 \times 10^{-2}$$

2.9 CHANGE IN DIRECTION OF A LINEAR ELEMENT

It is easy to calculate the change in the orientation of a linear element resulting from the deformation of the solid body. Let PQ be the element of length Δs, with direction cosines n_x, n_y and n_z. After deformation, the element becomes $P'Q'$ of length $\Delta s'$, with direction cosines n'_x, n'_y and n'_z. If u_x, u_y, u_z are the displacement components of point P, then the displacement components of point Q are.

$$u_x + \Delta u_x, \qquad u_y + \Delta u_y, \qquad u_z + \Delta u_z$$

where Δu_x, Δu_y and Δu_z are given by Eq. (2.7a)–(2.7c).
From Eq. (2.12), remembering that in the linear range $E_{PQ} = \varepsilon_{PQ}$,

$$\Delta s' = \Delta s\,(1 + \varepsilon_{PQ}) \tag{2.28}$$

The coordinates of P, Q, P' and Q' are as follows:

P: (x, y, z)

Q: $(x + \Delta x, y + \Delta y, z + \Delta z)$

P': $(x + u_x, y + u_y, z + u_z)$

Q': $(x + \Delta x + u_x + \Delta u_x, y + \Delta y + u_y + \Delta u_y, z + \Delta z + u_z + \Delta u_z)$

Hence,

$$n_x = \frac{\Delta x}{\Delta s}, \qquad n_y = \frac{\Delta y}{\Delta s}, \qquad n_z = \frac{\Delta z}{\Delta s}$$

$$n'_x = \frac{\Delta x + \Delta u_x}{\Delta s'}, \quad n'_y = \frac{\Delta y + \Delta u_y}{\Delta s'}, \quad n_z = \frac{\Delta z + \Delta u_z}{\Delta s'}$$

Substituting for $\Delta s'$ from Eq. (2.28) and for Δu_x, Δu_y, Δu_z from Eq. (2.7a)–(2.7c)

$$n'_x = \frac{1}{1+\varepsilon_{PQ}}\left[\left(1+\frac{\partial u_x}{\partial x}\right)n_x + \frac{\partial u_x}{\partial y}n_y + \frac{\partial u_x}{\partial z}n_z\right]$$

$$n'_y = \frac{1}{1+\varepsilon_{PQ}}\left[\frac{\partial u_y}{\partial x}n_x + \left(1+\frac{\partial u_y}{\partial y}\right)n_y + \frac{\partial u_y}{\partial z}n_z\right] \qquad (2.29)$$

$$n'_z = \frac{1}{1+\varepsilon_{PQ}}\left[\frac{\partial u_z}{\partial x}n_x + \frac{\partial u_z}{\partial y}n_y + \left(1+\frac{\partial u_z}{\partial z}\right)n_z\right]$$

The value of ε_{PQ} is obtained using Eq. (2.20).

2.10 CUBICAL DILATATION

Consider a point A with coordinates (x, y, z) and a neighbouring point B with coordinates $(x + \Delta x, y + \Delta y, z + \Delta z)$. After deformation, the points A and B move to A' and B' with coordinates

$$A' : (x + u_x, y + u_y, z + u_z)$$
$$B' : (x + \Delta x + u_x + \Delta u_x, y + \Delta y + u_y + \Delta u_y, z + \Delta z + u_z + \Delta u_z)$$

where u_x, u_y and u_z are the components of diplacements of point A, and from Eqs (2.7a)–(2.7c)

$$\Delta u_x = \frac{\partial u_x}{\partial x}\Delta x + \frac{\partial u_x}{\partial y}\Delta y + \frac{\partial u_x}{\partial z}\Delta z$$

$$\Delta u_y = \frac{\partial u_y}{\partial x}\Delta x + \frac{\partial u_y}{\partial y}\Delta y + \frac{\partial u_y}{\partial z}\Delta z$$

$$\Delta u_z = \frac{\partial u_z}{\partial x}\Delta x + \frac{\partial u_z}{\partial y}\Delta y + \frac{\partial u_z}{\partial z}\Delta z$$

The displaced segement $A'B'$ will have the following components along the x, y and z axes:

$$x \text{ axis: } \Delta x + \Delta u_x = \left(1+\frac{\partial u_x}{\partial x}\right)\Delta x + \frac{\partial u_x}{\partial y}\Delta y + \frac{\partial u_x}{\partial z}\Delta z$$

$$y \text{ axis: } \Delta y + \Delta u_y = \frac{\partial u_y}{\partial x}\Delta x + \left(1+\frac{\partial u_y}{\partial y}\right)\Delta y + \frac{\partial u_y}{\partial z}\Delta z \qquad (2.30)$$

$$z \text{ axis: } \Delta z + \Delta u_z = \frac{\partial u_z}{\partial x}\Delta x + \frac{\partial u_z}{\partial y}\Delta y + \left(1+\frac{\partial u_z}{\partial z}\right)\Delta z$$

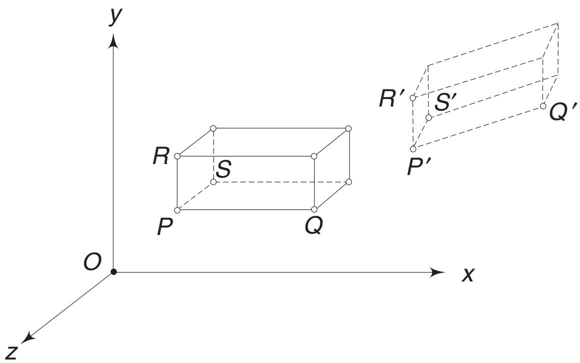

Fig. 2.5 *Deformation of right parallelepiped*

Consider now an infinitesimal rectangular parallelepiped with sides Δx, Δy and Δz (Fig. 2.5). When the body undergoes deformation, the right parallelepiped *PQRS* becomes an oblique parallelepiped *P′Q′R′S′*.

Identifying *PQ* of Fig. 2.5 with *AB* of Eqs (2.30), one has $\Delta y = \Delta z = 0$. Then, from Eqs (2.30) the projections of *P′Q′* will be

$$\text{along } x \text{ axis: } \left(1+\frac{\partial u_x}{\partial x}\right)\Delta x$$

$$\text{along } y \text{ axis: } \frac{\partial u_y}{\partial x}\Delta x$$

$$\text{along } z \text{ axis: } \frac{\partial u_z}{\partial x}\Delta x$$

Hence, one can successively identify *AB* with *PQ* ($\Delta y = \Delta z = 0$), *PR* ($\Delta x = \Delta z = 0$), *PS* ($\Delta x = \Delta y = 0$) and get the components of *P′Q′*, *P′R′* and *P′S′* along the *x*, *y* and *z* axes as

	P′Q′	*P′R′*	*P′S′*
x axis:	$\left(1+\frac{\partial u_x}{\partial x}\right)\Delta x$	$\frac{\partial u_x}{\partial y}\Delta y$	$\frac{\partial u_x}{\partial z}\Delta z$
y axis:	$\frac{\partial u_y}{\partial x}\Delta x$	$\left(1+\frac{\partial u_y}{\partial y}\right)\Delta y$	$\frac{\partial u_y}{\partial z}\Delta z$
z axis:	$\frac{\partial u_z}{\partial x}\Delta x$	$\frac{\partial u_z}{\partial y}\Delta y$	$\left(1+\frac{\partial u_z}{\partial z}\right)\Delta z$

The volume of the right parallelepiped before deformation is equal to $V = \Delta x\ \Delta y\ \Delta z$. The volume of the deformed parallelepiped is obtained from the well-known formula from analytic geometry as

$$V' = V + \Delta V = D \cdot \Delta x\ \Delta y\ \Delta z$$

where D is the following determinant:

$$D = \begin{vmatrix} \left(1+\dfrac{\partial u_x}{\partial x}\right) & \dfrac{\partial u_x}{\partial y} & \dfrac{\partial u_x}{\partial z} \\ \dfrac{\partial u_y}{\partial x} & \left(1+\dfrac{\partial u_y}{\partial y}\right) & \dfrac{\partial u_y}{\partial z} \\ \dfrac{\partial u_z}{\partial x} & \dfrac{\partial u_z}{\partial y} & \left(1+\dfrac{\partial u_z}{\partial z}\right) \end{vmatrix} \tag{2.31}$$

If we assume that the strains are small quantities such that their squares and products can be negelected, the above determinant becomes

$$D \approx 1 + \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}$$

$$= 1 + \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} \tag{2.32}$$

Hence, the new volume according to the linear strain theory will be

$$V' = V + \Delta V = (1 + \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz})\,\Delta x\,\Delta y\,\Delta z \tag{2.33}$$

The volumetric strain is defined as

$$\Delta = \frac{\Delta V}{V} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} \tag{2.34}$$

Therefore, according to the linear theory, the volumetric strain, also known as cubical dilatation, is equal to the sum of three linear strains.

Example 2.5 *The following state of strain exists at a point P*

$$[\varepsilon_{ij}] = \begin{bmatrix} 0.02 & -0.04 & 0 \\ -0.04 & 0.06 & 0.02 \\ 0 & -0.02 & 0 \end{bmatrix}$$

In the direction PQ having direction cosines $n_x = 0.6$, $n_y = 0$ *and* $n_z = 0.8$, *determine* ε_{PQ}.

Solution From Eq. (2.20)

$$\varepsilon_{PQ} = 0.02\,(0.36) + 0.06\,(0) + 0\,(0.64) - 0.04\,(0) - 0.02\,(0) + 0\,(0.48)$$
$$= 0.007$$

Example 2.6 *In Example 2.5, what is the cubical dilatation at point P?*

Solution From Eq. (2.34)

$$\Delta = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$$
$$= 0.02 + 0.06 + 0 = 0.08$$

2.11 CHANGE IN THE ANGLE BETWEEN TWO LINE ELEMENTS

Let PQ be a line element with direction cosines n_{x1}, n_{y1}, n_{z1} and PR be another line element with direction cosines n_{x2}, n_{y2}, n_{z2}, (Fig. 2.6). Let θ be the angle between the two line elements before deformation. After deformation, the line segments become $P'Q'$ and $P'R'$ with an included angle θ'. We can determine θ' easily from the results obtained in Sec. 2.9.

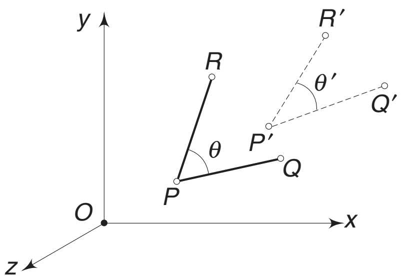

Fig. 2.6 *Change in angle between two line segments*

From analytical geometry

$$\cos\ \theta' = n'_{x1}\, n'_{x2} + n'_{y1}\, n'_{y2} + n'_{z1}\, n'_{z2}$$

The values of $n'_{x1}\, n'_{y1}, n'_{z1}, n'_{x2}, n'_{y2}$ and n'_{z2} can be substituted from Eq. (2.29). Neglecting squares and products of small strain components.

$$\cos\ \theta' = \frac{1}{(1+\varepsilon_{PQ})(1+\varepsilon_{PR})}\,[(1 + 2\varepsilon_{xx})\, n_{x1}n_{x2} + (1 + 2\varepsilon_{yy})\, n_{y1}n_{y2} + (1 + 2\varepsilon_{zz})\, n_{z1}n_{z2} + \gamma_{xy}(n_{x1}n_{y2} + n_{x2}n_{y1}) + \gamma_{yz}(n_{y1}n_{z2} + n_{y2}n_{z1}) + \gamma_{zx}(n_{x1}n_{z2} + n_{x2}\, n_{z1})] \tag{2.35}$$

In particular, if the two line segments PQ and PR are at right angles to each other before strain, then after strain,

$$\cos\ \theta' = \frac{1}{(1+\varepsilon_{PQ})(1+\varepsilon_{PR})}\,[2\varepsilon_{xx}\, n_{x1}\, n_{x2} + 2\varepsilon_{yy}\, n_{y1}\, n_{y2} + 2\varepsilon_{zz}\, n_{z1}\, n_{z2} + \gamma_{xy}\,(n_{x1}n_{y2} + n_{x2}n_{y1}) + \gamma_{yz}(n_{y1}n_{z2} + n_{y2}n_{z1}) + \gamma_{zx}(n_{x1}\, n_{z2} + n_{x2}\, n_{z1})] \tag{2.36a}$$

Now $(90° - \theta')$ represents the change in the initial right angle. If this is denoted by α, then

$$\theta' = 90° - \alpha \tag{2.36b}$$

or
$$\cos\ \theta' = \cos\,(90° - \alpha) = \sin\,\alpha \approx \alpha \tag{2.36c}$$

since α is small. Therefore Eq. (2.36a) gives the shear strain α between PQ and PR. If we represent the directions of PQ and PR at P by x' and y' axes, then

$\gamma_{x'y'}$ at $P = \cos\,\theta'$ = expression given in Eqs (2.36a), (2.36b) and (2.36c)

Example 2.7 *The displacement field for a body is given by*

$$\boldsymbol{u} = k(x^2 + y)\boldsymbol{i} + k(y + z)\boldsymbol{j} + k(x^2 + 2z^2)\boldsymbol{k} \quad \text{where } k = 10^{-3}$$

At a point P(2, 2, 3), consider two line segments PQ and PR having the following direction cosines before deformation

$$PQ : n_{x1} = n_{y1} = n_{z1} = \frac{1}{\sqrt{3}}, \qquad PR: n_{x2} = n_{y2} = \frac{1}{\sqrt{2}}, \qquad n_{z2} = 0$$

Determine the angle between the two segments before and after deformation.

Solution Before deformation, the angle θ between PQ and PR is

$$\cos\,\theta = n_{x1}\,n_{x2} + n_{y1}n_{y2} + n_{z1}n_{z2} = \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{6}} = 0.8165$$

$$\therefore \qquad \theta = 35.3°$$

The strain components at P after deformation are

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = 2kx = 4k, \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y} = k, \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z} = 4kz = 12k$$

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = k, \quad \gamma_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} = k, \quad \gamma_{zx} = \frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z} = 4k$$

The linear strains in directions PQ and PR are from Eq. (2.20)

$$\varepsilon_{PQ} = k\left[\left(4\times\frac{1}{3}\right) + \frac{1}{3} + \left(12\times\frac{1}{3}\right) + \left(1\times\frac{1}{3}\right) + \left(1\times\frac{1}{3}\right) + \left(4\times\frac{1}{3}\right)\right] = \frac{23}{3}k$$

$$\varepsilon_{PR} = k\left[\left(4\times\frac{1}{2}\right) + \left(1\times\frac{1}{2}\right) + (12\times 0) + \left(1\times\frac{1}{2}\right) + 0 + 0\right] = 3k$$

After deformation, the angle beteween $P'Q'$ and $P'R'$ is from Eq. (2.35)

$$\cos\,\theta' = \frac{1}{(1+23/3k)\,(1+3k)}\left[(1+8k)\,\frac{1}{\sqrt{6}} + (1+2k)\,\frac{1}{\sqrt{6}} + 0\right.$$

$$\left. + \left(\frac{1}{\sqrt{6}} + \frac{1}{\sqrt{6}}\right)k + \left(0 + \frac{1}{\sqrt{6}}\right)k + \left(0 + \frac{1}{\sqrt{6}}\right)4k\right]$$

$$= 0.8144 \quad \text{and} \quad \theta = 35.5°$$

2.12 PRINCIPAL AXES OF STRAIN AND PRINCIPAL STRAINS

It was shown in Sec. 2.5 that when a displacement field is defined at a point P, the relative extension (i.e. strain) at P in the direction PQ is given by Eq. (2.20) as

$$\varepsilon_{PQ} = \varepsilon_{xx}n_x^2 + \varepsilon_{yy}n_y^2 + \varepsilon_{zz}n_z^2 + \gamma_{xy}n_xn_y + \gamma_{yz}n_yn_z + \gamma_{xz}n_xn_z$$

As the values of n_x, n_y and n_z change, we get different values of strain ε_{PQ}. Now we ask ourselves the following questions:

What is the direction $(n_x,\ n_y,\ n_z)$ along which the strain is an extremum (i.e. maximum or minimum) and what is the corresponding extremum value?

According to calculus, in order to find the maximum or the minimum, we would have to equate,

$$\partial\varepsilon_{PQ}/\partial n_x, \qquad \partial\varepsilon_{PQ}/\partial n_y, \qquad \partial\varepsilon_{PQ}/\partial n_z,$$

to zero, if n_x, n_y and n_z were all independent. However, n_x, n_y and n_z are not all independent since they are related by the condition

$$n_x^2 + n_y^2 + n_z^2 = 1 \tag{2.37}$$

Taking n_x and n_y as independent and differentiating Eq. (2.37) with respect to n_x and n_y we get

$$\begin{aligned} 2n_x + 2n_z \frac{\partial n_z}{\partial n_x} &= 0 \\ 2n_y + 2n_z \frac{\partial n_z}{\partial n_y} &= 0 \end{aligned} \tag{2.38}$$

Differentiating ε_{PQ} with respect to n_x and n_y and equating them to zero for extremum

$$0 = 2n_x\varepsilon_{xx} + n_y\gamma_{xy} + n_z\gamma_{zx} + \frac{\partial n_z}{\partial n_x}\,(n_x\gamma_{zx} + n_y\gamma_{zy} + 2n_z\varepsilon_{zz})$$

$$0 = 2n_y\varepsilon_{yy} + n_x\gamma_{xy} + n_z\gamma_{yz} + \frac{\partial n_z}{\partial n_y}\,(n_x\gamma_{zx} + n_y\gamma_{zy} + 2n_z\varepsilon_{zz})$$

Substituting for $\partial n_z/\partial n_x$ and $\partial n_z/\partial n_y$ from Eqs (2.38),

$$\frac{2n_x\varepsilon_{xx} + n_y\gamma_{xy} + n_z\gamma_{zx}}{n_x} = \frac{n_x\gamma_{zx} + n_y\gamma_{zy} + 2n_z\varepsilon_{zz}}{n_z}$$

$$\frac{2n_y\varepsilon_{yy} + n_x\gamma_{xy} + n_z\gamma_{yz}}{n_y} = \frac{n_x\gamma_{zx} + n_y\gamma_{zy} + 2n_z\varepsilon_{zz}}{n_z}$$

Denoting the right-hand side expression in the above two equations by 2ε and rearranging,

$$2\varepsilon_{xx}n_x + \gamma_{xy}n_y + \gamma_{xz}n_z - 2\varepsilon n_x = 0 \tag{2.39a}$$

$$\gamma_{xy}n_x + 2\varepsilon_{yy}n_y + \gamma_{yz}n_z - 2\,\varepsilon n_y = 0 \tag{2.39b}$$

and

$$\gamma_{zx}n_x + \gamma_{zy}n_y + 2\varepsilon_{zz}n_z - 2\varepsilon n_z = 0 \tag{2.39c}$$

One can solve Eqs (2.39a)–(2.39c) to get the values of n_x, n_y and n_z, which determine the direction along which the relative extension is an extremum. Let us assume that this direction has been determined. Multiplying the first equation by n_x, second by n_y and the third by n_z and adding them, we get

$$2(\varepsilon_{xx}n_x^2 + \varepsilon_{yy}n_y^2 + \varepsilon_{zz}n_z^2 + \gamma_{xy}n_xn_y + \gamma_{yz}n_yn_z + \gamma_{zx}n_zn_x) = 2\varepsilon(n_x^2 + n_y^2 + n_z^2)$$

If we impose the condition $n_x^2 + n_y^2 + n_z^2 = 1$, the right-hand side becomes equal to 2ε. From Eq. (2.20), the left-hand side is the expression for $2\varepsilon_{PQ}$. Therefore

$$\varepsilon_{PQ} = \varepsilon$$

This means that in Eqs (2.39a)–(2.39c) the values of n_x, n_y and n_z determine the direction along which the relative extension is an extremum and further, the value of ε is equal to this extremum. Equations (2.39a)–(2.39c) can be written as

$$\begin{aligned} (\varepsilon_{xx} - \varepsilon)n_x + \frac{1}{2}\gamma_{xy}n_y + \frac{1}{2}\gamma_{xz}n_z &= 0 \\ \frac{1}{2}\gamma_{yx}n_x + (\varepsilon_{yy} - \varepsilon)n_y + \frac{1}{2}\gamma_{yz}n_z &= 0 \\ \frac{1}{2}\gamma_{zx}n_x + \frac{1}{2}\gamma_{zy}n_y + (\varepsilon_{zz} - \varepsilon)n_z &= 0 \end{aligned} \tag{2.40a}$$

If we adopt the notation given in Eq. (2.22), i.e. put

$$\frac{1}{2}\gamma_{xy} = e_{xy}, \qquad \frac{1}{2}\gamma_{yz} = e_{yz}, \qquad \frac{1}{2}\gamma_{zx} = e_{zx}$$

then Eqs (2.40a) can be written as

$$\begin{aligned} (\varepsilon_{xx} - \varepsilon)n_x + e_{xy}n_y + e_{xz}n_z &= 0 \\ e_{yx}n_x + (\varepsilon_{yy} - \varepsilon)n_y + e_{yz}n_z &= 0 \\ e_{zx}n_x + e_{zy}n_y + (\varepsilon_{zz} - \varepsilon)n_z &= 0 \end{aligned} \tag{2.40b}$$

The above set of equations is homogeneous in n_x, n_y and n_z. For the existence of a non-trivial solution, the determinant of its coefficient must be equal to zero, i.e.

$$\begin{vmatrix} (\varepsilon_{xx} - \varepsilon) & e_{xy} & e_{xz} \\ e_{yx} & (\varepsilon_{yy} - \varepsilon) & e_{yz} \\ e_{zx} & e_{zy} & (\varepsilon_{zz} - \varepsilon) \end{vmatrix} = 0 \tag{2.41}$$

Expanding the determinant, we get

$$\varepsilon^3 - J_1\varepsilon^2 + J_2\varepsilon - J_3 = 0 \tag{2.42}$$

where

$$J_1 = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} \tag{2.43}$$

$$J_2 = \begin{vmatrix} \varepsilon_{xx} & e_{xy} \\ e_{yx} & \varepsilon_{yy} \end{vmatrix} + \begin{vmatrix} \varepsilon_{yy} & e_{yz} \\ e_{zy} & \varepsilon_{zz} \end{vmatrix} + \begin{vmatrix} \varepsilon_{xx} & e_{xz} \\ e_{zx} & \varepsilon_{zz} \end{vmatrix} \tag{2.44}$$

$$J_3 = \begin{vmatrix} \varepsilon_{xx} & e_{xy} & e_{xz} \\ e_{yx} & \varepsilon_{yy} & e_{yz} \\ e_{zx} & e_{zy} & \varepsilon_{zz} \end{vmatrix} \tag{2.45}$$

It is important to observe that J_2 and J_3 involve e_{xy}, e_{yz} and e_{zx} not γ_{xy}, γ_{yz} and γ_{zx}. Equations (2.41)–(2.45) are all similar to Eqs (1.8), (1.9), (1.12), (1.13) and (1.14). The problem posed and its analysis are similar to the analysis of principal stress axes and principal stresses. The results of Sec. 1.10–1.15 can be applied to the case of strain.

For a given state of strain at point P, if the relative extension (i.e. strain) ε is an extremum in a direction $\boldsymbol{n}$, then ε is the principal strain at P and $\boldsymbol{n}$ is the principal strain direction associated with ε.

In every state of strain there exist at least three mutually perpendicular principal axes and at most three distinct principal strains. The principal strains ε_1, ε_2 and ε_3, are the roots of the cubic equation.

$$\varepsilon^3 - J_1\varepsilon^2 + J_2\varepsilon - J_3 = 0 \tag{2.46}$$

where J_1, J_2, J_3 are the first, second and third invariants of strain. The principal directions associated with ε_1, ε_2 and ε_3 are obtained by substituting ε_i ($i = 1, 2, 3$) in the following equations and solving for n_x, n_y and n_z.

$$\begin{aligned} (\varepsilon_{xx} - \varepsilon_i)n_x + e_{xy}n_y + e_{xz}n_z &= 0 \\ e_{xy}n_x + (\varepsilon_{yy} - \varepsilon_i)n_y + e_{yz}n_z &= 0 \\ n_x^2 + n_y^2 + n_z^2 &= 1 \end{aligned} \tag{2.47}$$