If ε_1, ε_2 and ε_3 are distinct, then the axes of $\boldsymbol{n}_1$, $\boldsymbol{n}_2$ and $\boldsymbol{n}_3$ are unique and mutually peprendicular. If, say $\varepsilon_1 = \varepsilon_2 \neq \varepsilon_3$, then the axis of $\boldsymbol{n}_3$ is unique and every direction perpendicular to $\boldsymbol{n}_3$ is a principal direction associated with $\varepsilon_1 = \varepsilon_2$.

If $\varepsilon_1 = \varepsilon_2 = \varepsilon_3$, then every direction is a principal direction.

Example 2.8 *The displacement field in micro units for a body is given by*

$$\boldsymbol{u} = (x^2 + y)\boldsymbol{i} + (3 + z)\boldsymbol{j} + (x^2 + 2y)\boldsymbol{k}$$

Determine the principal strains at (3, 1, –2) and the direction of the minimum principal strain.

Solution The displacement components in micro units are,

$$u_x = x^2 + y, \quad u_y = 3 + z, \quad u_z = x^2 + 2y.$$

The rectangular strain components are

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = 2x, \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y} = 0, \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z} = 0$$

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = 1, \quad \gamma_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} = 3, \quad \gamma_{zx} = \frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z} = 2x$$

At point (3, 1, –2) the strain components are therefore,

$$\varepsilon_{xx} = 6, \quad \varepsilon_{yy} = 0, \quad \varepsilon_{zz} = 0$$
$$\gamma_{xy} = 1, \quad \gamma_{yz} = 3, \quad \gamma_{zx} = 6$$

The strain invariants from Eqs (2.43) – (2.45) are

$$J_1 = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = 6$$

$$J_2 = \begin{vmatrix} 6 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{vmatrix} + \begin{vmatrix} 0 & \frac{3}{2} \\ \frac{3}{2} & 0 \end{vmatrix} + \begin{vmatrix} 6 & 3 \\ 3 & 0 \end{vmatrix} = -\frac{23}{2}$$

Note that J_2 and J_3 involve $e_{xy} = \frac{1}{2}\gamma_{xy}, \quad e_{yz} = \frac{1}{2}\gamma_{yz}, \quad e_{zx} = \frac{1}{2}\gamma_{zx}$

$$J_3 = \begin{vmatrix} 6 & \frac{1}{2} & 3 \\ \frac{1}{2} & 0 & \frac{3}{2} \\ 3 & \frac{3}{2} & 0 \end{vmatrix} = -9$$

The cubic from Eq. (2.46) is

$$\varepsilon^3 - 6\varepsilon^2 - \frac{23}{2}\varepsilon + 9 = 0$$

Following the standard method suggested in Sec. 1.15

$$a = \frac{1}{3}\left(-\frac{69}{2} - 36\right) = -\frac{47}{2}$$

$$b = \frac{1}{27}(-432 - 621 + 243) = -30$$

$$\cos\ \phi = -\frac{-30}{2 \times \sqrt{-a^3/27}} = 0.684$$

$$\therefore \qquad \phi = 46°48'$$

$$g = 2\sqrt{-a/3} = 5.6$$

The principal strains in micro units are

$$\varepsilon_1 = g \cos \phi/3 + 2 = +7.39$$

$$\varepsilon_2 = g \cos (\phi/3 + 120°) + 2 = -2$$

$$\varepsilon_3 = g \cos (\phi/3 + 240°) + 2 = +0.61$$

As a check, the first invariant J_1 is

$$\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 7.39 - 2 + 0.61 = 6$$

The second invariant J_2 is

$$\varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3 + \varepsilon_3\varepsilon_1 = -14.78 - 1.22 + 4.51 = -11.49$$

The third invariant J_3 is

$$\varepsilon_1\varepsilon_2\varepsilon_3 = 7.39 \times 2 \times 0.61 = -9$$

These agree with the earlier values.

The minimum principal strain is –2. For this, from Eq. (2.47)

$$(6 + 2)\, n_x + \frac{1}{2} n_y + 3n_z = 0$$

$$\frac{1}{2} n_x + 2n_y + \frac{3}{2} n_z = 0$$

$$n_x^2 + n_y^2 + n_z^2 = 1$$

The solutions are $n_x = 0.267$, $n_y = 0.534$ and $n_z = -0.801$.

Example 2.9 *For the state of strain given in Example 2.5, determine the principal strains and the directions of the maximum and minimum principal strains.*

Solution From the strain matrix given, the invariants are

$$J_1 = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = 0.02 + 0.06 + 0 = 0.08$$

$$J_2 = \begin{vmatrix} 0.02 & -0.02 \\ -0.02 & 0.06 \end{vmatrix} + \begin{vmatrix} 0.06 & -0.01 \\ -0.01 & 0 \end{vmatrix} + \begin{vmatrix} 0.02 & 0 \\ 0 & 0 \end{vmatrix}$$

$$= (0.0012 - 0.0004) + (-0.0001) + 0 = 0.0007$$

$$J_3 = \begin{vmatrix} 0.02 & -0.02 & 0 \\ -0.02 & 0.06 & -0.01 \\ 0 & -0.01 & 0 \end{vmatrix} = 0.02\,(-0.0001) + 0 + 0 = -0.000002$$

The cubic equation is

$$\varepsilon^3 - 0.08\varepsilon^2 + 0.0007\varepsilon + 0.000002 = 0$$

Following the standard procedure described in Sec. 1.15, one can determine the principal strains. However, observing that the constant J_3 in the cubic is very small, one can ignore it and write the cubic as

$$\varepsilon^2 - 0.08\varepsilon^2 + 0.0007\varepsilon = 0$$

One of the solutions obviously is $\varepsilon = 0$. For the other two solutions (ε not equal to zero), dividing by ε

$$\varepsilon^2 - 0.08\,\varepsilon + 0.0007 = 0$$

The solutions of this quadratic equation are

$$\varepsilon = 0.4 \pm 0.035, \text{ i.e. } \quad 0.075 \quad \text{and} \quad 0.005$$

Rearranging such that $\varepsilon_1 \geq \varepsilon_2 \geq \varepsilon_3$, the principal strains are

$$\varepsilon_1 = 0.07, \quad \varepsilon_2 = 0.01, \quad \varepsilon_3 = 0$$

As a check:

$$J_1 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0.07 + 0.01 = 0.08$$

$$J_2 = \varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_2 + \varepsilon_3\varepsilon_1 = (0.07 \times 0.01) = 0.0007$$

$$J_3 = \varepsilon_1\varepsilon_2\varepsilon_3 = 0 \qquad \text{(This was assumed as zero)}$$

Hence, these values agree with their previous values. To determine the direction of $\varepsilon_1 = 0.07$, from Eqs (2.47)

$$(0.02 - 0.07)\, n_x - 0.02 n_y = 0$$

$$-\,0.02 n_x + (0.06 - 0.07)\, n_y - 0.01 n_z = 0$$

$$n_x^2 + n_y^2 + n_z^2 = 1$$

The solutions are $n_x = 0.44, \quad n_y = -0.176$ and $n_z = 0.88$.
Similarly, for $\varepsilon_3 = 0$, from Eqs (2.47)

$$0.02 n_x - 0.02 n_y = 0$$

$$-0.02 n_x + 0.06 n_y - 0.01 n_z = 0$$

$$n_x^2 + n_y^2 + n_z^2 = 1$$

The solutions are $n_x = n_y = 0.236$ and $n_z = 0.944$.

2.13 PLANE STATE OF STRAIN

If, in a given state of strain, there exists a coordinate system *Oxyz*, such that for this system

$$\varepsilon_{zz} = 0, \quad \gamma_{yz} = 0, \quad \gamma_{zx} = 0 \tag{2.48}$$

then the state is said to have a plane state of strain parallel to the *xy* plane. The non-vanishing strain components are ε_{xx}, ε_{yy} and γ_{xy}.

If *PQ* is a line element in this *xy* plane, with direction cosines n_x, n_y, then the relative extension or the strain ε_{PQ} is obtained from Eq. (2.20) as

$$\varepsilon_{PQ} = \varepsilon_{xx}\, n_x^2 + \varepsilon_{yy}\, n_y^2 + \gamma_{xy}\, n_x\, n_y$$

or if *PQ* makes an angle θ with the *x* axis, then

$$\varepsilon_{PQ} = \varepsilon_{xx} \cos^2\theta + \varepsilon_{yy} \sin^2\theta + \frac{1}{2}\gamma_{xy} \sin 2\theta \tag{2.49}$$

If ε_1 and ε_2 are the principal strains, then

$$\varepsilon_1, \varepsilon_2, = \frac{\varepsilon_{xx} + \varepsilon_{yy}}{2} \pm \left[\left(\frac{\varepsilon_{xx} - \varepsilon_{yy}}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2\right]^{1/2} \tag{2.50}$$

Note that $\varepsilon_3 = \varepsilon_{zz}$ is also a principal strain. The principal strain axes make angles ϕ and $\phi + 90°$ with the x axis, such that

$$\tan 2\phi = \frac{\gamma_{xy}}{\varepsilon_{xx} - \varepsilon_{yy}} \tag{2.51}$$

The discussions and conclusions will be identical with the analysis of stress if we use ε_{xx}, ε_{yy}, and ε_{zz} in place of σ_x, σ_y and σ_z respectively, and $e_{xy} = \frac{1}{2}\gamma_{xy}$, $e_{yz} = \frac{1}{2}\gamma_{yz}$, $e_{zx} = \frac{1}{2}\gamma_{zx}$ in place of τ_{xy}, τ_{yz} and τ_{zx} respectively.

2.14 THE PRINCIPAL AXES OF STRAIN REMAIN ORTHOGONAL AFTER STRAIN

Let PQ be one of the principal extensions or strain axes with direction cosines n_{x1}, n_{y1} and n_{z1}. Then according to Eqs (2.40b)

$$(\varepsilon_{xx} - \varepsilon_1)n_{x1} + e_{xy}n_{y1} + e_{xz}n_{z1} = 0$$
$$e_{xy}n_{x1} + (\varepsilon_{yy} - \varepsilon_1)n_{y1} + e_{yz}n_{z1} = 0$$
$$e_{xz}n_{x1} + e_{yz}n_{y1} + (\varepsilon_{zz} - \varepsilon_1)n_{z1} = 0$$

Let n_{x2}, n_{y2} and n_{z2} be the direction cosines of a line PR, perpendicular to PQ before strain. Therefore,

$$n_{x1}n_{x2} + n_{y1}n_{y2} + n_{z1}n_{z2} = 0$$

Multiplying Eq. (2.40b), given above, by n_{x2}, n_{y2} and n_{z2} respectively and adding, we get,

$$\varepsilon_{xx}n_{x1}n_{x2} + \varepsilon_{yy}n_{y1}n_{y2} + \varepsilon_{zz}n_{z1}n_{z2} + e_{xy}(n_{x1}n_{y2} + n_{y1}n_{x2}) + e_{yz}(n_{y1}n_{z2} + n_{y2}n_{z1}) + e_{zx}(n_{x1}n_{z2} + n_{x2}n_{z1}) = 0$$

Multiplying by 2 and putting

$$2e_{xy} = \gamma_{xy}, \qquad 2e_{yz} = \gamma_{yz}, \qquad 2e_{zx} = \gamma_{zx}$$

we get

$$2\varepsilon_{xx}n_{x1}n_{x2} + 2\varepsilon_{yy}n_{y1}n_{y2} + 2\varepsilon_{zz}n_{z1}nz_2 + \gamma_{xy}(n_{x1}n_{y2} + n_{y1}n_{x2}) + \gamma_{yz}(n_{y1}n_{z2} + n_{y2}n_{z1}) + \gamma_{zx}(n_{x1}n_{z2} + n_{z1}n_{x2}) = 0$$

Comparing the above with Eq. (2.36a), we get

$$\cos\theta'(1 + \varepsilon_{PQ})(1 + \varepsilon_{PR}) = 0$$

where θ' is the new angle between PQ and PR after strain.

Since ε_{PQ} and ε_{PR} are quite general, to satisfy the equation, $\theta' = 90°$, i.e. the line segments remain perpendicular after strain also. Since PR is an arbitrary perpendicular line to the principal axis PQ, every line perpendicular to PQ before strain remains perpendicular after strain. In particular, PR can be the second principal axis of strain.

Repeating the above steps, if PS is the third principal axis of strain perpendicular to PQ and PR, it remains perpendicular after strain also. Therefore, at point P, we can identify a small rectangular element, with faces normal to the principal axes of strain, that will remain rectangular after strain also.

2.15 PLANE STRAINS IN POLAR COORDINATES

We now consider displacements and deformations of a two-dimensional radial element in polar coordinates. The polar coordinates of a point a are r and θ. The radial and circumferential displacements are denoted by u_r and u_θ. Consider an elementary radial element $abcd$, as shown in Fig. 2.7.

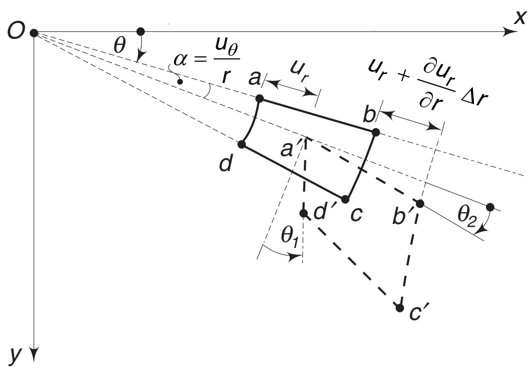

Fig. 2.7 *Displacement components of a radial element*

Point a with coordinates (r, θ) gets displaced after deformation to position a' with coordinates $(r + u_r, \theta + \alpha)$. The neighbouring point $b(r + \Delta r, \theta)$ gets moved to b' with coordinates

$$\left(r + \Delta r + u_r + \frac{\partial u_r}{\partial r}\Delta r, \theta + \alpha + \frac{\partial \alpha}{\partial r}\Delta r\right)$$

The length of $a'b'$ is therefore

$$\Delta r + \frac{\partial u_r}{\partial r}\Delta r$$

The radial strain ε_r is therefore

$$\varepsilon_r = \frac{\partial u_r}{\partial r} \tag{2.52}$$

The circumferential strain ε_θ is caused in two ways. If the element $abcd$ undergoes a purely radial displacement, then the length $ad = r\,\Delta\theta$ changes to $(r + u_r)\Delta\theta$. The strain due to this radial movement alone is

$$\frac{u_r \Delta\theta}{r\Delta\theta} = \frac{u_r}{r}$$

In addition to this, the point d moves circumferentially to d' through the distance

$$u_\theta + \frac{\partial u_\theta}{\partial \theta}\Delta\theta$$

Since point a moves circumferentially through u_θ, the change in ad is $\frac{\partial u_\theta}{\partial \theta}\Delta\theta$. The strain due to this part is

$$\frac{\partial u_\theta}{\partial \theta}\frac{\Delta\theta}{r\Delta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}$$

The total circumferential strain is therefore

$$\varepsilon_\theta = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} \tag{2.53}$$

To determine the shear strain we observe the following:

The circumferential displacement of a is u_θ, whereas that of b is $u_\theta + \dfrac{\partial u_\theta}{\partial r}\Delta r$. The magnitude of θ_2 is

$$\left[\left(u_\theta + \frac{\partial u_\theta}{\partial r}\Delta r\right) - \alpha\left(r + \Delta r\right)\right]\frac{1}{\Delta r}$$

But $$\alpha = \frac{u_\theta}{r}.$$

Hence, $$\theta_2 = \left(u_\theta + \frac{\partial u_\theta}{\partial r}\Delta r - u_\theta - \frac{u_\theta}{r}\Delta r\right)\frac{1}{\Delta r}$$

$$= \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}$$

Similarly, the radial displacement of a is u_r, whereas that of d is $u_r + \dfrac{\partial u_r}{\partial \theta}\Delta\theta$.

Hence,

$$\theta_1 = \frac{1}{r\Delta\theta}\left[\left(u_r + \frac{\partial u_r}{\partial \theta}\Delta_\theta\right) - u_r\right]$$

$$= \frac{1}{r}\frac{\partial u_r}{\partial \theta}$$

Hence, the shear strain $\gamma_{r\theta}$ is

$$\gamma_{r\theta} = \theta_1 + \theta_2 = \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \tag{2.54}$$

2.16 COMPATIBILITY CONDITIONS

It was observed that the displacement of a point in a solid body can be represented by a displacement vector $\boldsymbol{u}$, which has components,

$$u_x,\ u_y,\ u_z.$$

along the three axes x, y and z respectively. The deformation at a point is specified by the six strain components,

$$\varepsilon_{xx},\ \varepsilon_{yy},\ \varepsilon_{zz},\ \gamma_{xy},\ \gamma_{yz} \text{ and } \gamma_{zx}.$$

The three displacement components and the six rectangular strain components are related by the six strain displacement relations of Cauchy, given by Eqs (2.18) and (2.19). The determination of the six strain components from the three displacement functions is straightforward since it involves only differentiation. However, the reverse operation, i.e. determination of the three displacement functions from the six strain components is more complicated since it involves integrating six equations to obtain three functions. One may expect, therefore, that all the six strain components cannot be prescribed arbitrarily and there must exist certain relations among these. The total number of these relations are six and they fall into two groups.

First group: We have

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \qquad \varepsilon_{yy} = \frac{\partial u_y}{\partial y}, \qquad \gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$

Differentiate the first two of the above equations as follows:

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} = \frac{\partial^3 u_x}{\partial x \partial y^2} = \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial u_x}{\partial y}\right)$$

$$\frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^3 u_y}{\partial y \partial x^2} = \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial u_y}{\partial x}\right)$$

Adding these two, we get

$$\frac{\partial^2}{\partial x \partial y}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

i.e.

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

Similarly, by considering ε_{yy}, ε_{zz} and γ_{yz}, and ε_{zz}, ε_{xx} and γ_{zx}, we get two more conditions. This leads us to the first group of conditions.

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

$$\frac{\partial^2 \varepsilon_{yy}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} \qquad (2.55)$$

$$\frac{\partial^2 \varepsilon_{zz}}{\partial x^2} + \frac{\partial^2 \varepsilon_{xx}}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x}$$

Second group: This group establishes the conditions among the shear strains. We have

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$

$$\gamma_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}$$

$$\gamma_{xz} = \frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}$$

Differentiating

$$\frac{\partial \gamma_{xy}}{\partial z} = \frac{\partial^2 u_x}{\partial z \partial y} + \frac{\partial^2 u_y}{\partial z \partial x}$$

$$\frac{\partial \gamma_{yz}}{\partial x} = \frac{\partial^2 u_y}{\partial x \partial z} + \frac{\partial^2 u_z}{\partial x \partial y}$$

$$\frac{\partial \gamma_{zx}}{\partial y} = \frac{\partial^2 u_z}{\partial x \partial y} + \frac{\partial^2 u_x}{\partial y \partial z}$$

Adding the last two equations and subtracting the first

$$\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z} = 2\frac{\partial^2 u_z}{\partial x\, \partial y}$$

Differentiating the above equation once more with respect to z and observing that

$$\frac{\partial^3 u_z}{\partial x \partial y \partial z} = \frac{\partial^2 \varepsilon_{zz}}{\partial x\, \partial y}$$

we get,

$$\frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right) = 2\frac{\partial^3 u_z}{\partial x\, \partial y\, \partial z} = 2\frac{\partial^2 \varepsilon_{zz}}{\partial x\, \partial y}$$

This is one of the required relations of the second group. By a cyclic change of the letters we get the other two equations. Collecting all equations, the six strain compatibility relations are

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x\, \partial y} \tag{2.56a}$$

$$\frac{\partial^2 \varepsilon_{yy}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y\, \partial z} \tag{2.56b}$$

$$\frac{\partial^2 \varepsilon_{zz}}{\partial x^2} + \frac{\partial^2 \varepsilon_{xx}}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z\, \partial x} \tag{2.56c}$$

$$\frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right) = 2\frac{\partial^2 \varepsilon_{zz}}{\partial x\, \partial y} \tag{2.56d}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} - \frac{\partial \gamma_{yz}}{\partial x}\right) = 2\frac{\partial^2 \varepsilon_{xx}}{\partial y\, \partial z} \tag{2.56e}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y}\right) = 2\frac{\partial^2 \varepsilon_{yy}}{\partial x\, \partial z} \tag{2.56f}$$

The above six equations are called Saint-Venant's equations of compatibility. We can give a geometrical interpretation to the above equations. For this purpose, imagine an elastic body cut into small parallelepipeds and give each of them the deformation defined by the six strain components. It is easy to conceive that if the components of strain are not connected by certain relations, it is impossible to make a continuous deformed solid from individual deformed parallelepipeds. Saint-Venant's compatibility relations furnish these conditions. Hence, these equations are also known as continuity equations.

Example 2.10 *For a circular rod subjected to a torque (Fig. 2.8), the displacement components at any point (x, y, z) are obtained as*

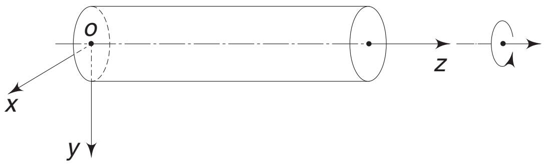

Fig. 2.8 *Example 2.8*

$$u_x = -\tau yz + ay + bz + c$$
$$u_y = \tau xz - ax + ez + f$$
$$u_z = -bx - ey + k$$

where a, b, c, e, f and k are constants, and τ is the shear stress.

(i) *Select the constants a, b, c, e, f, k such that the end section z = 0 is fixed in the following manner:*
   (a) *Point o has no displacement.*
   (b) *The element Δz of the axis rotates neither in the plane xoz nor in the plane yoz*
   (c) *The element Δy of the axis does not rotate in the plane xoy.*

(ii) *Determine the strain components.*

(iii) *Verify whether these strain components satisfy the compatibility conditions.*

Solution

(i) Since point '*o*' does not have any displacement

$$u_x = c = 0, \qquad u_y = f = 0, \qquad u_z = k = 0$$

The displacements of a point Δz from '*o*' are

$$\frac{\partial u_x}{\partial z}\Delta z, \quad \frac{\partial u_y}{\partial z}\Delta z \quad \text{and} \quad \frac{\partial u_z}{\partial z}\Delta z$$

Similarly, the displacements of a point Δ*y* from '*o*' are

$$\frac{\partial u_x}{\partial y}\Delta y, \quad \frac{\partial u_y}{\partial y}\Delta y \quad \text{and} \quad \frac{\partial u_z}{\partial y}\Delta y$$

Hence, according to condition (b)

$$\frac{\partial u_y}{\partial z}\Delta z = 0 \quad \text{and,} \quad \frac{\partial u_x}{\partial z}\Delta z = 0$$

and according to condition (c)

$$\frac{\partial u_x}{\partial y}\Delta y = 0$$

Applying these requirements

$$\frac{\partial u_y}{\partial z} \text{ at 'o' is } e \text{ and hence, } e = 0$$

$$\frac{\partial u_x}{\partial z} \text{ at 'o' is } b \text{ and hence, } b = 0$$

$$\frac{\partial u_x}{\partial y} \text{ at 'o' is } a \text{ and hence, } a = 0$$

Consequently, the displacement components are

$$u_x = -\tau yz, \qquad u_y = \tau xz \qquad \text{and} \qquad u_z = 0$$

(ii) The strain components are

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = 0, \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y} = 0, \quad \varepsilon_{zz} = 0;$$

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = -\tau z + \tau z = 0$$

$$\gamma_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial x} = \tau x$$

$$\gamma_{zx} = \frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z} = -\tau y$$

(iii) Since the strain components are linear in x, y and z, the Saint-Venant's compatibility requirements are automatically satisfied.

2.17 STRAIN DEVIATOR AND ITS INVARIANTS

Similar to the analysis of stress, we can resolve the e_{ij} matrix into a spherical (i.e. isotoropic) and a deviatoric part. The e_{ij} matrix is

$$\left[e_{ij}\right] = \begin{bmatrix} \varepsilon_{xx} & e_{xy} & e_{xz} \\ e_{xy} & \varepsilon_{yy} & e_{yz} \\ e_{xz} & e_{yz} & \varepsilon_{zz} \end{bmatrix}$$

This can be resolved into two parts as

$$\left[e_{ij}\right] = \begin{bmatrix} \varepsilon_{xx} - e & e_{xy} & e_{xz} \\ e_{xy} & \varepsilon_{yy} - e & e_{yz} \\ e_{xz} & e_{yz} & \varepsilon_{zz} - e \end{bmatrix} + \begin{bmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e \end{bmatrix} \tag{2.57}$$

where
$$e = \frac{1}{3}\left(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}\right) \tag{2.58}$$

represents the mean elongation at a given point. The second matrix on the right-hand side of Eq. (2.57) is the spherical part of the strain matrix. The first matrix represents the deviatoric part or the strain deviator. If an isolated element of the body is subjected to the strain deviator only, then according to Eq. (2.34), the volumetric strain is equal to

$$\begin{aligned} \frac{\Delta V}{V} &= (\varepsilon_{xx} - e) + (\varepsilon_{yy} - e) + (\varepsilon_{zz} - e) \\ &= \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} - 3e \\ &= 0 \end{aligned} \tag{2.59}$$

This means that an element subjected to deviatoric strain undergoes pure deformation without a change in volume. Hence, this part is also known as the pure shear part of the strain matrix. This discussion is analogous to that made in Sec. 1.22. The spherical part of the strain matrix, i.e. the second matrix on the right-hand side of Eq. (2.57) is an isotropic state of strain. It is called isotropic because when a body is subjected to this particular state of strain, then every direction is a principal strain direction, with a strain of magnitude e, according to Eq. (2.20). A sphere subjected to this state of strain will uniformally expand or contract and remain spherical.

Consider the invariants of the strain deviator. These are constructed in the same way as the invariants of the stress and strain matrices with an appropriate replacement of notations.

(i) Linear invariant is zero since

$$J_1' = (\varepsilon_{xx} - e) + (\varepsilon_{yy} - e) + (\varepsilon_{zz} - e) = 0 \tag{2.60}$$

(ii) Quadratic invariant is

$$J_2' = \begin{bmatrix} \varepsilon_{xx} - e & e_{xy} \\ e_{xy} & \varepsilon_{yy} - e \end{bmatrix} + \begin{bmatrix} \varepsilon_{yy} - e & e_{yz} \\ e_{yz} & \varepsilon_{zz} - e \end{bmatrix} + \begin{bmatrix} \varepsilon_{xx} - e & e_{xz} \\ e_{xz} & \varepsilon_{zz} - e \end{bmatrix}$$

$$= -\frac{1}{6}\left[\left(\varepsilon_{xx} - \varepsilon_{yy}\right)^2 + \left(\varepsilon_{yy} - \varepsilon_{zz}\right)^2 + \left(\varepsilon_{zz} - \varepsilon_{xx}\right)^2 \right. \tag{2.61}$$

$$\left. + 6\left(e_{xy} + e_{yx} + e_{zx}\right)^2\right]$$

(iii) Cubic invariant is

$$J_3' = \begin{vmatrix} \varepsilon_{xx} - e & e_{xy} & e_{xz} \\ e_{xy} & \varepsilon_{yy} - e & e_{yz} \\ e_{xz} & e_{zy} & \varepsilon_{zz} - e \end{vmatrix} \tag{2.62}$$

The second and third invariants of the deviatoric strain matrix describe the two types of distortions that an isolated element undergoes when subjected to the given strain matrix e_{ij}.

Problems

2.1 The displacement field for a body is given by

$$\boldsymbol{u} = (x^2 + y)\boldsymbol{i} + (3 + z)\boldsymbol{j} + (x^2 + 2y)\boldsymbol{k}$$

Write down the displacement gradient matrix at point (2, 3, 1).

$$\left[Ans.\ \begin{bmatrix} 4 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 2 & 0 \end{bmatrix} \right]$$

2.2 The displacement field for a body is given by

$$\boldsymbol{u} = [(x^2 + y^2 + 2)\boldsymbol{i} + (3x + 4y^2)\boldsymbol{j} + (2x^3 + 4z)\boldsymbol{k}]10^{-4}$$

What is the displaced position of a point originally at (1, 2, 3)?

[*Ans.* (1.0007, 2.0019, 3.0014)]

2.3 For the displacement field given in Problem 2.2, what are the strain components at (1, 2, 3). Use only linear terms.

$$\left[\begin{array}{l} Ans. \quad \varepsilon_{xx} = 0.0002,\ \varepsilon_{yy} = 0.0016,\ \varepsilon_{zz} = 0.0004 \\ \qquad\quad \gamma_{xy} = 0.0007,\ \gamma_{yz} = 0,\ \gamma_{zx} = 0.0006 \end{array} \right]$$

2.4 What are the strain acomponents for Problem 2.3, if non-linear terms are also included?

$$\left[\begin{array}{l} Ans. \quad E_{xx} = 2p + 24.5p^2, \quad E_{yy} = 16p + 136p^2,\ E_{zz} = 4p + 8p^2 \\ \qquad\quad E_{xy} = 7p + 56p^2, \quad E_{yz} = 0,\ E_{zx} = 6p + 24p^2 \text{ where } p = 10^{-4} \end{array} \right]$$

2.5 If the displacement field is given by

$$u_x = kxy, \qquad u_y = kxy, \qquad u_z = 2k(x + y)z$$

where k is a constant small enough to ensure applicability of the small deformation theory,

(a) write down the strain matrix

(b) what is the strain in the direction $n_x = n_y = n_z = 1/\sqrt{3}$?

$$\left[\begin{array}{l} Ans. \quad (a) \left[\varepsilon_{ij}\right] = k \begin{bmatrix} y & x+y & 2z \\ x+y & x & 2z \\ 2z & 2z & 2(x+y) \end{bmatrix} \\ \qquad\quad (b)\ \varepsilon_{PQ} = \dfrac{4k}{3}(x + y + z) \end{array} \right]$$

2.6 The displacement field is given by

$$u_x = k(x^2 + 2z), \qquad u_y = k(4x + 2y^2 + z), \qquad u_z = 4kz^2$$

k is a very small constant. What are the strains at (2, 2, 3) in directions

(a) $n_x = 0,\ n_y = 1/\sqrt{2}\ ,\ n_z = 1/\sqrt{2}$

(b) $n_x = 1,\ n_y = n_z = 0$

(c) $n_x = 0.6,\ n_y = 0,\ n_z = 0.8$

$$\left[Ans. \quad (a)\ \frac{33}{2}k,\ (b)\ 4k,\ (c)\ 17.76k \right]$$

2.7 For the displacement field given in Problem 2.6, with $k = 0.001$, determine the change in angle between two line segments PQ and PR at $P(2, 2, 3)$ having direction cosines before deformation as

(a) PQ: $\quad n_{x1} = 0,\ \ n_{y1} = n_{z1} = \dfrac{1}{\sqrt{2}}$

PR: $\quad n_{x2} = 1,\ \ n_{y2} = n_{z2} = 0$

(b) PQ: $\quad n_{x1} = 0,\ n_{y1} = n_{z1} = \dfrac{1}{\sqrt{2}}$

PR: $\quad n_{x2} = 0.6,\ n_{y2} = 0,\ n_{z2} = 0.8$

$$\left[\begin{array}{l} Ans. \quad (a) \quad 90^\circ - 89.8^\circ = 0.2^\circ \\ \qquad\quad\ \ (b) \quad 55.5^\circ - 50.7^\circ = 4.8^\circ \end{array} \right]$$

2.8. The rectangular components of a small strain at a point is given by the following matrix. Determine the principal strains and the direction of the maximum unit strain (i.e. ε_{max}).

$$\left[\varepsilon_{ij}\right] = p \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -4 \\ 0 & -4 & 3 \end{bmatrix} \text{ where } p = 10^{-4}$$

$$\left[\begin{array}{l} Ans.\ \varepsilon_1 = 4p,\ \varepsilon_2 = p,\ \varepsilon_3 = -p \\ \text{for } \varepsilon_1 : n_x = 0,\ n_y = 0.447,\ n_z = 0.894 \\ \text{for } \varepsilon_2 : n_x = 1,\ n_y = n_z = 0 \\ \text{for } \varepsilon_3 : n_x = 0,\ n_y = 0.894,\ n_z = 0.447 \end{array}\right]$$

2.9 For the following plane strain distribution, verify whether the compatibility condition is satisfied:

$$\varepsilon_{xx} = 3x^2y, \qquad \varepsilon_{yy} = 4y^2x + 10^{-2}, \qquad \gamma_{xy} = 2xy + 2x^3$$

[*Ans.* Not satisfied]

2.10 Verify whether the following strain field satisfies the equations of compatibility. p is a constant:

$$\varepsilon_{xx} = py, \qquad \varepsilon_{yy} = px, \qquad \varepsilon_{zz} = 2p(x + y)$$
$$\gamma_{xy} = p(x + y), \qquad \varepsilon_{yz} = 2pz, \qquad \varepsilon_{zx} = 2pz$$

[*Ans.* Yes]

2.11 State the conditions under which the following is a possible system of strains:

$$\varepsilon_{xx} = a + b(x^2 + y^2)\, x^4 + y^4, \qquad \gamma_{yz} = 0$$
$$\varepsilon_{yy} = \alpha + \beta(x^2 + y^2) + x^4 + y^4, \qquad \gamma_{zx} = 0$$
$$\gamma_{xy} = A + Bxy\,(x^2 + y^2 - c^2), \qquad \varepsilon_{zz} = 0$$

[*Ans.* $B = 4;\ b + \beta + 2c^2 = 0$]

2.12 Given the following system of strains

$$\varepsilon_{xx} = 5 + x^2 + y^2 + x^4 + y^4$$
$$\varepsilon_{yy} = 6 + 3x^2 + 3y^2 + x^4 + y^4$$
$$\gamma_{xy} = 10 + 4xy\,(x^2 + y^2 + 2)$$
$$\varepsilon_{zz} = \gamma_{yz} = \gamma_{zx} = 0$$

determine whether the above strain field is possible. If it is possible, determine the displacement components in terms of x and y, assuming that $u_x = u_y = 0$ and $\omega_{xy} = 0$ at the origin.

$$\left[\begin{array}{ll} Ans.\ \text{It is possible.} & u_x = 5x + \frac{1}{3}x^3 + xy^2 + \frac{1}{5}x^5 + xy^4 + cy \\ & u_y = 6y + 3x^2y + y^3 + x^4y + \frac{1}{5}y^5 + cx \end{array}\right]$$

2.13 For the state of strain given in Problem 2.12, write down the spherical part and the deviatoric part and determine the volumetric strain.

$$\left[\begin{array}{l} Ans.\ \text{Components of spherical part are} \\ e = \frac{1}{3}[11 + 4(x^2 + y^2) + 2(x^4 + y^4)] \\ \text{Volumetric strain} = 11 + 4(x^2 + y^2) + 2(x^4 + y^4) \end{array}\right]$$

Appendix

On Compatibility Conditions

It was stated in Sec. 2.16 that the six strain components e_{ij} (i.e., $e_{xx} = \varepsilon_{xx}$, $e_{yy} = \varepsilon_{yy}$, $e_{zz} = \varepsilon_{zz}$, $e_{xy} = \frac{1}{2}\gamma_{xy}$, $e_{yz} = \frac{1}{2}\gamma_{zy}$, $e_{zx} = \frac{1}{2}\gamma_{zx}$) should satisfy certain necessary conditions for the existence of single-valued, continuous displacement functions, and these were called compatibility conditions. In a two-dimensional case, these conditions reduce to

$$\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} = 2\frac{\partial^2 e_{xy}}{\partial x \partial y}$$

Generally, these equations are obtained by differentiating the expressions for e_{xx}, e_{yy}, e_{xy}, and showing their equivalence in the above manner. However, their requirement for the existence of single-value displacement is not shown. In this section, this aspect will be treated for the plane case.

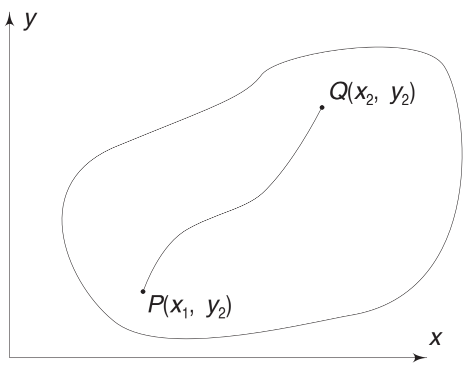

Fig. A.1 *Continuous curve connecting P and Q in a simply connected body.*

Let P($x_1 - y_1$) be some point in a simply connected region at which the displacement (u_x°, u_y°) are known. We try to determine the displacements (u_x, u_y) at another point Q in terms of the known functions e_{xx}, e_{yy}, e_{xy}, ω_{xy} by means of a line integral over a simple continuous curve C joining the points P and Q.

Consider the displacement u_x

$$u_x(x_2, y_2) = u_x^\circ + \int_P^Q du_x \tag{A.1}$$

Since,
$$du_x = \frac{\partial u_x}{\partial x}dx + \frac{\partial u_x}{\partial y}dy$$

$$u_x(x_2, y_2) = u_x^\circ + \int_P^Q \frac{\partial u_x}{\partial x}dx + \int_P^Q \frac{\partial u_x}{\partial y}dy$$

$$= u_x^\circ + \int_P^Q e_{xx}d_x + \int_P^Q \frac{\partial u_x}{\partial y}dy$$

Now, $$\frac{\partial u_x}{\partial y} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) + \frac{1}{2}\left(\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x}\right)$$

$= e_{xy} - \omega_{yx}$ from equations (2.22) and (2.25).

$$\therefore \quad u_x(x_2, y_2) = u_x^\circ + \int_P^Q e_{xx}dx + \int_P^Q e_{xy}dy - \int_P^Q \omega_{yx}dy \tag{A.2}$$

Integrating by parts, the last integral on the right-hand side

$$\int_P^Q \omega_{yx}dy = (y\omega_{yx})\Big|_P^Q - \int_P^Q y\,d(\omega_{yx})$$

$$= (y\omega_{yx})\Big|_P^Q - \int_P^Q y\left(\frac{\partial \omega_{yx}}{\partial x}dx + \frac{\partial \omega_{yx}}{\partial y}dy\right) \tag{A.3}$$

Substituting, Eq. (A.2) becomes

$$u_x(x_2, y_2) = u_x^\circ + \int_P^Q e_{xx}dx + \int_P^Q e_{xy}dx - (y\omega_{yx})\Big|_P^Q - \int_P^Q y\left(\frac{\partial \omega_{yx}}{\partial x}dx + \frac{\partial \omega_{yx}}{\partial y}dy\right) \tag{A.4}$$

Now consider the terms in the last integral on the right-hand side.

$$\frac{\partial \omega_{yx}}{\partial x} = \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x}\right)$$

$$= \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x}\right) + \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial u_x}{\partial x} - \frac{\partial u_x}{\partial x}\right)$$

adding and subtracting $\frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial u_x}{\partial x}\right)$.

Since the order of differentiation is immaterial.

$$\frac{\partial \omega_{yx}}{\partial x} = \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_x}{\partial x}\right) - \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}e_{xx} - \frac{\partial}{\partial x}e_{xy} \tag{A.5}$$

Similarly, $$\frac{\partial \omega_{xy}}{\partial y} = \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x}\right)$$

$$= \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x}\right) + \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial u_y}{\partial y} - \frac{\partial u_y}{\partial y}\right) \tag{A.6}$$

$$= \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) - \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial u_y}{\partial y} + \frac{\partial u_y}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}e_{xy} - \frac{\partial}{\partial x}e_{yy}$$

Substituting (A.5) and (A.6) in (A.4)

$$u_x(x_2, y_2) = u_x^\circ - (y\omega_{yx})\Big|_P^Q + \int_P^Q e_{xx}dx + \int_P^Q e_{xy}dy$$

$$-\int y\left[\left(\frac{\partial}{\partial y}e_{xx} - \frac{\partial}{\partial x}e_{xy}\right)dx + \left(\frac{\partial}{\partial y}e_{yx} - \frac{\partial}{\partial x}e_{yy}\right)dy\right]$$

Regrouping,

$$u_x(x_2, y_2) = u^\circ_x - (y\omega_{yx})\Big|_P^Q + \int_P^Q\left[e_{xx} - y\frac{\partial e_{xx}}{\partial y} + y\frac{\partial e_{xy}}{\partial x}\right]dx$$

$$+\int_P^Q\left[e_{xy} - y\frac{\partial e_{yx}}{\partial y} + y\frac{\partial e_{yy}}{\partial x}\right]dy \qquad \text{(A.7)}$$

Since the displacement is single–valued, the integral should be independent of the path of integration. This means that the integral is a perfect differential. This means

$$\frac{\partial}{\partial y}\left[e_{xx} - y\frac{\partial e_{xx}}{\partial y} + y\frac{\partial e_{xy}}{\partial x}\right] = \frac{\partial}{\partial x}\left[e_{xy} - y\frac{\partial e_{yx}}{\partial y} + y\frac{\partial e_{yy}}{\partial x}\right]$$

i.e., $$\frac{\partial e_{xx}}{\partial y} - \frac{\partial e_{xx}}{\partial y} - y\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial e_{xy}}{\partial x} + y\frac{\partial^2 e_{xy}}{\partial x\partial y} = \frac{\partial e_{xy}}{\partial x} - y\frac{\partial^2 e_{xy}}{\partial x\partial y} + y\frac{\partial^2 e_{yy}}{\partial x^2}$$

Since $e_{xy} = e_{yx}$, the above equation reduces to

$$\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} = 2\frac{\partial^2 e_{xy}}{\partial x\partial y} \qquad \text{(A.8)}$$

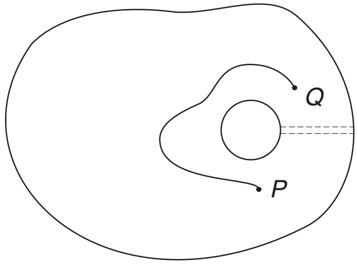

Fig. A.2 *Continuous curve connecting P and Q but not passing through the cut of multiply connected body*

An identical expression is obtained while considering the displacement u_y (x_2, y_2). Hence, the compatibility condition is a necessary and sufficient condition for the existence of single-valued displacement functions in simply connected bodies. For a multiply connected body, it is a necessary but not a sufficient condition. A multiply connected body can be made simply connected by a suitable cut. The displacement functions will then become single-valued when the path of integration does not pass through the cut.

Stress–Strain Relations for Linearly Elastic Solids

CHAPTER 3

3.1 INTRODUCTION

In the preceding two chapters we dealt with the state of stress at a point and the state of strain at a point. The strain components were related to the displacement components through six of Cauchy's strain-displacement relationships. In this chapter, the relationships between the stress and strain components will be established. Such equations are termed constitutive equations. They depend on the manner in which the material resists deformation.

The constitutive equations are mathematical descriptions of the physical phenomena based on experimental observations and established principles. Consequently, they are approximations of the true behavioural pattern, since an accurate mathematical representation of the physical phenomena would be too complicated and unworkable.

The constitutive equations describe the behaviour of a material, not the behaviour of a body. Therefore, the equations relate the state of stress at a point to the state of strain at the point.

3.2 GENERALISED STATEMENT OF HOOKE'S LAW

Consider a uniform cylindrical rod of diameter d subjected to a tensile force P. As is well known from experimental observations, when P is gradually increased from zero to some positive value, the length of the rod also increases. Based on experimental observations, it is postulated in elementary strength of materials that the axial stress σ is proportional to the axial strain ε up to a limit called the proportionality limit. The constant of proportionality is the Young's Modulus E, i.e.

$$\varepsilon = \frac{\sigma}{E} \quad \text{or} \quad \sigma = E\varepsilon \tag{3.1}$$

It is also well known that when the uniform rod elongates, its lateral dimensions, i.e. its diameter, decreases. In elementary strength of materials, the ratio of lateral strain to longitudinal strain was termed as Poisson's ratio ν. We now extend this information or knowledge to relate the six rectangular components of stress to the six rectangular components of strain. We assume that each of the six independent components of stress may be expressed as a linear function of the six components of strain and vice versa.

The mathematical expressions of this statement are the six stress–strain equations:

$$\begin{aligned}
\sigma_x &= a_{11}\varepsilon_{xx} + a_{12}\varepsilon_{yy} + a_{13}\varepsilon_{zz} + a_{14}\gamma_{xy} + a_{15}\gamma_{yz} + a_{16}\gamma_{zx} \\
\sigma_y &= a_{21}\varepsilon_{xx} + a_{22}\varepsilon_{yy} + a_{23}\varepsilon_{zz} + a_{24}\gamma_{xy} + a_{25}\gamma_{yz} + a_{26}\gamma_{zx} \\
\sigma_z &= a_{31}\varepsilon_{xx} + a_{32}\varepsilon_{yy} + a_{33}\varepsilon_{zz} + a_{34}\gamma_{xy} + a_{35}\gamma_{yz} + a_{36}\gamma_{zx} \\
\tau_{xy} &= a_{41}\varepsilon_{xx} + a_{42}\varepsilon_{yy} + a_{43}\varepsilon_{zz} + a_{44}\gamma_{xy} + a_{45}\gamma_{yz} + a_{46}\gamma_{zx} \\
\tau_{yz} &= a_{51}\varepsilon_{xx} + a_{52}\varepsilon_{yy} + a_{53}\varepsilon_{zz} + a_{54}\gamma_{xy} + a_{55}\gamma_{yz} + a_{56}\gamma_{zx} \\
\tau_{zx} &= a_{61}\varepsilon_{xx} + a_{62}\varepsilon_{yy} + a_{63}\varepsilon_{zz} + a_{64}\gamma_{xy} + a_{65}\gamma_{yz} + a_{66}\gamma_{zx}
\end{aligned} \tag{3.2}$$

Or conversely, six strain-stress equations of the type:

$$\varepsilon_{xx} = b_{11}\sigma_x + b_{12}\sigma_y + b_{13}\sigma_z + b_{14}\tau_{xy} + b_{15}\tau_{yz} + b_{16}\tau_{zx} \tag{3.3}$$

$\varepsilon_{yy} = \ldots$ etc

where a_{11}, a_{12}, b_{11}, b_{12}, . . . , are constants for a given material. Solving Eq. (3.2) as six simultaneous equations, one can get Eq. (3.3), and vice versa. For homogeneous, linearly elastic material, the six Eqs (3.2) or (3.3) are known as Generalised Hooke's Law. Whether we use the set given by Eq. (3.2) or that given by Eq. (3.3), 36 elastic constants are apparently involved.

3.3 STRESS–STRAIN RELATIONS FOR ISOTROPIC MATERIALS

We now make a further assumption that the ideal material we are dealing with has the same properties in all directions so far as the stress-strain relations are concerned. This means that the material we are dealing with is isotropic, i.e. it has no directional property.

Care must be taken to distinguish between the assumption of isotropy, which is a particular statement regarding the stress-strain properties at a given point, and that of homogeneity, which is a statement that the stress-strain properties, whatever they may be, are the same at all points. For example, timber of regular grain is homogeneous but not isotropic.

Assuming that the material is isotropic, one can show that only two independent elastic constants are involved in the generalised statement of Hooke's law. In Chapter 1, it was shown that at any point there are three faces (or planes) on which the resultant stresses are wholly normal, i.e. there are no shear stresses on these planes. These planes were termed the principal planes and the stresses on these planes the principal stresses. In Sec. 2.14, it was shown that at any point one can identify before strain, a small rectangular parallelepiped or a box which remains rectangular after strain. The normals to the faces of this box were called the principal axes of strain. Since in an isotropic material, a small rectangular box the faces of which are subjected to pure normal stresses, will remain rectangular after deformation (no asymmetrical deformation), the normal to these faces coincide with the principal strain axes. Hence, for an isotropic material, one can relate the principal stresses σ_1, σ_2, σ_3 with the three principal strains ε_1, ε_2 and ε_3 through suitable elastic constants. Let the axes *x*, *y* and *z* coincide with the principal stress and principal strain directions. For the principal stress σ_1 the equation becomes

$$\sigma_1 = a\varepsilon_1 + b\varepsilon_2 + c\varepsilon_3$$

where *a*, *b* and *c* are constants. But we observe that *b* and *c* should be equal since the effect of σ_1 in the directions of ε_2 and ε_3, which are both at right angles to σ_1, must be the same for an isotropic material. In other words, the effect of σ_1 in any direction transverse to it is the same in an isotropic material. Hence, for σ_1 the equation becomes

$$\begin{aligned}\sigma_1 &= a\varepsilon_1 + b(\varepsilon_2 + \varepsilon_3)\\ &= (a - b)\varepsilon_1 + b(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)\end{aligned}$$

by adding and subtracting $b\varepsilon_1$. But $(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$ is the first invariant of strain $\boldsymbol{J}_1$ or the cubical dilatation Δ. Denoting *b* by λ and $(a - b)$ by 2μ, the equation for σ_1 becomes

$$\sigma_1 = \lambda\Delta + 2\mu\varepsilon_1 \tag{3.4a}$$

Similarly, for σ_2 and σ_3 we get

$$\sigma_2 = \lambda\Delta + 2\mu\varepsilon_2 \tag{3.4b}$$

$$\sigma_3 = \lambda\Delta + 2\mu\varepsilon_3 \tag{3.4c}$$

The constants λ and μ are called Lame's coefficients. Thus, there are only two elastic constants involved in the relations between the principal stresses and principal strains for an isotropic material. As the next sections show, this can be extended to the relations between rectangular stress and strain components also.

3.4 MODULUS OF RIGIDITY

Let the co-ordinate axes *Ox*, *Oy*, *Oz* coincide with the principal stress axes. For an isotropic body, the principal strain axes will also be along *Ox*, *Oy*, *Oz*. Consider another frame of reference *Ox′*, *Oy′*, *Oz′*, such that the direction cosines of *Ox′* are n_{x1}, n_{y1}, n_{z1} and those of *Oy′* are n_{x2}, n_{y2}, n_{z2}. Since *Ox′* and *Oy′* are at right angles to each other.

$$n_{x1}n_{x2} + n_{y1}n_{y2} + n_{z1}n_{z2} = 0 \tag{3.5}$$

The normal stress $\sigma_{x'}$ and the shear stress $\tau_{x'y'}$ are obtained from Cauchy's formula, Eqs. (1.9). The resultant stress vector on the *x′* plane will have components as

$$\overset{x'}{T}_x = n_{x1}\sigma_1, \qquad \overset{x'}{T}_y = n_{y1}\sigma_2, \qquad \overset{x'}{T}_z = n_{z1}\sigma_3$$

These are the components in *x*, *y* and *z* directions. The normal stress on this *x′* plane is obtained as the sum of the projections of the components along the normal, i.e.

$$\sigma_n = \sigma_{x'} = n_{x1}^2\sigma_1 + n_{y1}^2\sigma_2 + n_{z1}^2\sigma_3 \tag{3.6a}$$

Similarly, the shear stress component on this *x′* plane in *y′* direction is obtained as the sum of the projections of the components in *y′* direction, which has direction cosines n_{x2}, n_{y2}, n_{z2}. Thus

$$\tau_{x'y'} = n_{x1}n_{x2}\sigma_1 + n_{y1}n_{y2}\sigma_2 + n_{z1}n_{z2}\sigma_3 \tag{3.6b}$$

On the same lines, if ε_1, ε_2 and ε_3 are the principal strains, which are also along x, y, z directions, the normal strain in x' direction, from Eq. (2.20), is

$$\varepsilon_{x'x'} = n_{x1}^2\varepsilon_1 + n_{y1}^2\varepsilon_2 + n_{z1}^2\varepsilon_3 \tag{3.7a}$$

The shear strain $\gamma_{x'y'}$ is obtained from Eq. (2.36c) as

$$\gamma_{x'y'} = \frac{1}{(1+\varepsilon_{x'})(1+\varepsilon_{y'})}\Big[2\big(n_{x1}n_{x2}\varepsilon_1 + n_{y1}n_{y2}\varepsilon_2 + n_{z1}n_{z2}\varepsilon_3\big)$$
$$+ n_{x1}n_{x2} + n_{y1}n_{y2} + n_{z1}n_{z2}\Big]$$

Using Eq. (3.5), and observing that $\varepsilon_{x'}$ and $\varepsilon_{y'}$ are small compared to unity in the denominator,

$$\gamma_{x'y'} = 2(n_{x1}n_{x2}\varepsilon_1 + n_{y1}n_{y2}\varepsilon_2 + n_{z1}n_{z2}\varepsilon_3) \tag{3.7b}$$

Substituting the values of σ_1, σ_2 and σ_3 from Eqs (3.4a)–(3.4c) into Eq. (3.6b)

$$\tau_{x'y'} = n_{x1}n_{x2}(\lambda\Delta + 2\mu\varepsilon_1) + n_{y1}n_{y2}(\lambda\Delta + 2\mu\varepsilon_2) + n_{z1}n_{z2}(\lambda\Delta + 2\mu\varepsilon_3)$$
$$= \lambda\Delta(n_{x1}n_{x2} + n_{y1}n_{y2} + n_{z1}n_{z2}) + 2\mu(n_{x1}n_{x2}\varepsilon_1 + n_{y1}n_{y2}\varepsilon_2 + n_{z1}n_{z2}\varepsilon_3)$$

Hence, from Eqs (3.5) and (3.7b)

$$\tau_{x'y'} = \mu\gamma_{x'y'} \tag{3.8}$$

Equation (3.8) relates the rectangular shear stress component $\tau_{x'y'}$ with the rectangular shear strain component $\gamma_{x'y'}$. Comparing this with the relation used in elementary strength of materials, one observes that μ is the modulus of rigidity, usually denoted by G.

By taking another axis Oz' with direction cosines n_{x3}, n_{y3} and n_{z3} and at right angles to Ox' and Oy' (so that $Ox'y'z'$ forms an orthogonal set of axes), one can get equations similar to (3.6a) and (3.6b) for the other rectangular stress components. Thus,

$$\sigma_{y'} = n_{x2}^2\sigma_1 + n_{y2}^2\sigma_2 + n_{z2}^2\sigma_3 \tag{3.9a}$$
$$\sigma_{z'} = n_{x3}^2\sigma_1 + n_{y3}^2\sigma_2 + n_{z3}^2\sigma_3 \tag{3.9b}$$
$$\tau_{y'z'} = n_{x2}n_{x3}\sigma_1 + n_{y2}n_{y3}\sigma_2 + n_{z2}n_{z3}\sigma_3 \tag{3.9c}$$
$$\tau_{z'x'} = n_{x3}n_{x1}\sigma_1 + n_{y3}n_{y1}\sigma_2 + n_{z3}n_{z1}\sigma_3 \tag{3.9d}$$

Similarly, following Eqs (3.7a) and (3.7b) for the other rectangular strain components, one gets

$$\varepsilon_{y'y'} = n_{x2}^2\varepsilon_1 + n_{y2}^2\varepsilon_2 + n_{z2}^2\varepsilon_3 \tag{3.10a}$$
$$\varepsilon_{z'z'} = n_{x3}^2\varepsilon_1 + n_{y3}^2\varepsilon_2 + n_{z3}^2\varepsilon_3 \tag{3.10b}$$
$$\gamma_{y'z'} = 2(n_{x2}n_{x3}\varepsilon_1 + n_{y2}n_{y3}\varepsilon_2 + n_{z2}n_{z3}\varepsilon_3) \tag{3.10c}$$
$$\gamma_{z'x'} = 2(n_{x3}n_{x1}\varepsilon_1 + n_{y3}n_{y1}\varepsilon_2 + n_{z3}n_{z1}\varepsilon_3) \tag{3.10d}$$

From Eqs (3.6a), (3.4a)–(3.4c) and (3.7a)

$$\sigma_{x'} = n_{x1}^2\sigma_1 + n_{y1}^2\sigma_2 + n_{z1}^2\sigma_3$$

$$= \lambda\Delta\left(n_{x1}^2 + n_{y1}^2 + n_{z1}^2\right) + 2\mu\left(\varepsilon_1 n_{x1}^2 + \varepsilon_2 n_{y1}^2 + \varepsilon_3 n_{z1}^2\right)$$

$$= \lambda\Delta + 2\mu\,\varepsilon_{x'x'} \tag{3.11a}$$

Similarly, one gets

$$\sigma_{y'} = \lambda\Delta + 2\mu\,\varepsilon_{y'y'} \tag{3.11b}$$

$$\sigma_{z'} = \lambda\Delta + 2\mu\,\varepsilon_{z'z'} \tag{3.11c}$$

Similar to Eq. (3.8),

$$\tau_{y'z'} = \mu\,\gamma_{y'z'} \tag{3.12a}$$

$$\tau_{x'z'} = \mu\,\gamma_{z'x'} \tag{3.12b}$$

Equations (3.11a)–(3.11c), (3.8) and (3.12a) and (3.12b) relate the six rectangular stress components to six rectangular strain components and in these only two elastic constants are involved. Therefore, the Hooke's law for an isotropic material will involve two independent elastic constants λ and μ (or G).

3.5 BULK MODULUS

Adding equations (3.11a)–(3.11c)

$$\sigma_{x'} + \sigma_{y'} + \sigma_{z'} = 3\lambda\Delta + 2\mu\left(\varepsilon_{x'x'} + \varepsilon_{y'y'} + \varepsilon_{z'z'}\right) \tag{3.13a}$$

Observing that

$$\sigma_{x'} + \sigma_{y'} + \sigma_{z'} = I_1 = \sigma_1 + \sigma_2 + \sigma_3 \qquad \text{(first invariant of stress)},$$

and

$$\varepsilon_{x'x'} + \varepsilon_{y'y'} + \varepsilon_{z'z'} = J_1 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \qquad \text{(first invariant of strain)},$$

Eq. (3.13a) can be written in several alternative forms as

$$\sigma_1 + \sigma_2 + \sigma_3 = (3\lambda + 2\mu)\Delta \tag{3.13b}$$

$$\sigma_{x'} + \sigma_{y'} + \sigma_{z'} = (3\lambda + 2\mu)\Delta \tag{3.13c}$$

$$I_1 = (3\lambda + 2\mu)J_1 \tag{3.13d}$$

Noting from Eq. (2.34) that Δ is the volumetric strain, the definition of bulk modulus K is

$$K = \frac{\text{pressure}}{\text{volumetric strain}} = \frac{p}{\Delta} \tag{3.14a}$$

If $\sigma_1 = \sigma_2 = \sigma_3 = p$, then from Eq. (3.13b)

$$3p = (3\lambda + 2\mu)\Delta$$

or

$$3\,\frac{p}{\Delta} = (3\lambda + 2\mu)$$

and from Eq. (3.14a)

$$K = \frac{1}{3}(3\lambda + 2\mu) \tag{3.14b}$$

Thus, the bulk modulus for an isotropic solid is related to Lame's constants through Eq. (3.14b).

3.6 YOUNG'S MODULUS AND POISSON'S RATIO

From Eq. (3.13b), we have

$$\Delta = \frac{\sigma_1 + \sigma_2 + \sigma_3}{(3\lambda + 2\mu)}$$

Substituting this in Eq. (3.4a)

$$\sigma_1 = \frac{\lambda}{(3\lambda + 2\mu)}(\sigma_1 + \sigma_2 + \sigma_3) + 2\mu\varepsilon_1$$

or

$$\varepsilon_1 = \frac{\lambda + \mu}{\mu(3\lambda + 2\mu)}\left[\sigma_1 - \frac{\lambda}{2(\lambda + \mu)}(\sigma_2 + \sigma_3)\right] \tag{3.15}$$

From elementary strength of materials

$$\varepsilon_1 = \frac{1}{E}[\sigma_1 - \nu(\sigma_2 + \sigma_3)]$$

where E is Young's modulus, and ν is Poisson's ratio. Comparing this with Eq. (3.15),

$$E = \frac{\mu(3\lambda + 2\mu)}{(\lambda + \mu)}; \qquad \nu = \frac{\lambda}{2(\lambda + \mu)} \tag{3.16}$$

3.7 RELATIONS BETWEEN THE ELASTIC CONSTANTS

In elementary strength of materials, we are familiar with Young's modulus E, Poisson's ratio ν, shear modulus or modulus of rigidity G and bulk modulus K. Among these, only two are independent, and E and ν are generally taken as the independent constants. The other two, namely, G and K, are expressed as

$$G = \frac{E}{2(1+\nu)}, \qquad K = \frac{E}{3(1-2\nu)} \tag{3.17}$$

It has been shown in this chapter, that for an isotropic material, the 36 elastic constants involved in the Generalised Hooke's law, can be reduced to two independent elastic constants. These two elastic constants are Lame's coefficients λ and μ. The second coefficient μ is the same as the rigidity modulus G. In terms of these, the other elastic constants can be expressed as

$$E = \frac{\mu(3\lambda + 2\mu)}{(\lambda + \mu)}, \qquad \nu = \frac{\lambda}{2(\lambda + \mu)}$$

$$K = \frac{(3\lambda + 2\mu)}{3}, \qquad G \equiv \mu, \qquad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \tag{3.18}$$

It should be observed from Eq. (3.17) that for the bulk modulus to be positive, the value of Poisson's ratio ν cannot exceed 1/2. This is the upper limit for ν. For $\nu = 1/2$,

$$3G = E \quad \text{and} \quad K = \infty$$

A material having Poisson's ratio equal to 1/2 is known as an incompressible material, since the volumetric strain for such an isotropic material is zero.

For easy reference one can collect the equations relating stresses and strains that have been obtained so far.

(i) In terms of principal stresses and principal strains:

$$\begin{aligned}\sigma_1 &= \lambda\Delta + 2\mu\varepsilon_1\\ \sigma_2 &= \lambda\Delta + 2\mu\varepsilon_2\\ \sigma_3 &= \lambda\Delta + 2\mu\varepsilon_3\end{aligned} \tag{3.19}$$

where $\Delta = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = J_1$.

$$\begin{aligned}\varepsilon_1 &= \frac{\lambda+\mu}{\mu(3\lambda+2\mu)}\left[\sigma_1 - \frac{\lambda}{2(\lambda+\mu)}(\sigma_2+\sigma_3)\right]\\ \varepsilon_2 &= \frac{\lambda+\mu}{\mu(3\lambda+2\mu)}\left[\sigma_2 - \frac{\lambda}{2(\lambda+\mu)}(\sigma_3+\sigma_1)\right]\\ \varepsilon_3 &= \frac{\lambda+\mu}{\mu(3\lambda+2\mu)}\left[\sigma_3 - \frac{\lambda}{2(\lambda+\mu)}(\sigma_1+\sigma_2)\right]\end{aligned} \tag{3.20}$$

(ii) In terms of rectangular stress and strain components referred to an orthogonal coordinate system *Oxyz*:

$$\begin{aligned}\sigma_x &= \lambda\Delta + 2\mu\varepsilon_{xx}\\ \sigma_y &= \lambda\Delta + 2\mu\varepsilon_{yy}\\ \sigma_z &= \lambda\Delta + 2\mu\varepsilon_{zz}\end{aligned} \tag{3.21a}$$

where $\Delta = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = J_1$.

$$\tau_{xy} = \mu\gamma_{xy}, \qquad \tau_{yz} = \mu\gamma_{yz}, \qquad \tau_{zx} = \mu\gamma_{zx} \tag{3.21b}$$

$$\begin{aligned}\varepsilon_{xx} &= \frac{\lambda+\mu}{\mu(3\lambda+2\mu)}\left[\sigma_x - \frac{\lambda}{2(\lambda+\mu)}(\sigma_y+\sigma_z)\right]\\ \varepsilon_{yy} &= \frac{\lambda+\mu}{\mu(3\lambda+2\mu)}\left[\sigma_y - \frac{\lambda}{2(\lambda+\mu)}(\sigma_z+\sigma_x)\right]\\ \varepsilon_{zz} &= \frac{\lambda+\mu}{\mu(3\lambda+2\mu)}\left[\sigma_z - \frac{\lambda}{2(\lambda+\mu)}(\sigma_x+\sigma_y)\right]\end{aligned} \tag{3.22a}$$

$$\gamma_{xy} = \frac{1}{\mu}\tau_{xy}, \qquad \gamma_{yz} = \frac{1}{\mu}\tau_{yz}, \qquad \gamma_{zx} = \frac{1}{\mu}\tau_{zx} \tag{3.22b}$$

In the preceeding sets of equations, λ and μ are Lame's constants. In terms of the more familiar elastic constants E and ν, the stress-strain relations are:

(iii) with $\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = J_1 = \Delta$,

$$\begin{aligned}\sigma_x &= \frac{E}{(1+\nu)}\left[\frac{\nu}{(1-2\nu)}\Delta + \varepsilon_{xx}\right]\\ &= \lambda J_1 + 2G\varepsilon_{xx}\end{aligned}$$

$$\sigma_y = \frac{E}{(1+\nu)}\left[\frac{\nu}{(1-2\nu)}\Delta + \varepsilon_{yy}\right] \tag{3.23a}$$

$$= \lambda J_1 + 2G\varepsilon_{yy}$$

$$\sigma_z = \frac{E}{(1+\nu)}\left[\frac{\nu}{(1-2\nu)}\Delta + \varepsilon_{zz}\right]$$

$$= \lambda J_1 + 2G\varepsilon_{zz}$$

$$\tau_{xy} = G\gamma_{xy}, \qquad \tau_{yz} = G\gamma_{yz}, \qquad \tau_{xx} = G\gamma_{zx} \tag{3.23b}$$

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right]$$

$$\varepsilon_{yy} = \frac{1}{E}\left[\sigma_y - \nu\left(\sigma_z + \sigma_x\right)\right] \tag{3.24a}$$

$$\varepsilon_{zz} = \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right]$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy}, \qquad \gamma_{yz} = \frac{1}{G}\tau_{yz}, \qquad \gamma_{zx} = \frac{1}{G}\tau_{zx} \tag{3.24b}$$

3.8 DISPLACEMENT EQUATIONS OF EQUILIBRIUM

In Chapter 1, it was shown that if a solid body is in equilibrium, the six rectangular stress components have to satisfy the three equations of equilibrium. In this chapter, we have shown how to relate the stress components to the strain components using the stress-strain relations. Hence, stress equations of equilibrium can be converted to strain equations of equilibrium. Further, in Chapter 2, the strain components were related to the displacement components. Therefore, the strain equations of equilibrium can be converted to displacement equations of equilibrium. In this section, this result will be derived.

The first equation from Eq. (1.65) is

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0$$

For an isotropic material

$$\sigma_x = \lambda\Delta + 2\mu\varepsilon_{xx}; \qquad \tau_{xy} = \mu\gamma_{xy}; \qquad \tau_{xz} = \mu\gamma_{xz}$$

Hence, the above equation becomes

$$\lambda\frac{\partial \Delta}{\partial x} + \mu\left(2\frac{\partial \varepsilon_{xx}}{\partial x} + \frac{\partial \gamma_{xy}}{\partial y} + \frac{\partial \gamma_{xz}}{\partial z}\right) = 0$$

From Cauchy's strain-displacement relations

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \qquad \gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}, \qquad \gamma_{zx} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}$$

Substituting these

$$\lambda \frac{\partial \Delta}{\partial x} + \mu \left(2\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_y}{\partial x \partial y} + \frac{\partial^2 u_x}{\partial z^2} + \frac{\partial^2 u_z}{\partial x \partial z} \right) = 0$$

or $$\lambda \frac{\partial \Delta}{\partial x} + \mu \left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2} \right) + \mu \left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_y}{\partial x \partial y} + \frac{\partial^2 u_z}{\partial x \partial z} \right) = 0$$

or $$\lambda \frac{\partial \Delta}{\partial x} + \mu \left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2} \right) + \mu \frac{\partial}{\partial x} \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right) = 0$$

Observing that

$$\Delta = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}$$

$$(\lambda + \mu) \frac{\partial}{\partial x} \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right) + \mu \left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2} \right) = 0$$

This is one of the displacement equations of equilibrium. Using the notation

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

the displacement equation of equilibrium becomes

$$(\lambda + \mu) \frac{\partial \Delta}{\partial x} + \mu \nabla^2 u_x = 0 \tag{3.25a}$$

Similarly, from the second and third equations of equilibrium, one gets

$$(\lambda + \mu) \frac{\partial \Delta}{\partial y} + \mu \nabla^2 u_y = 0 \tag{3.25b}$$

$$(\lambda + \mu) \frac{\partial \Delta}{\partial z} + \mu \nabla^2 u_z = 0$$

These are known as Lame's displacement equations of equilibrium. They involve a synthesis of the analysis of stress, analysis of strain and the relations between stresses and strains. These equations represent the mechanical, geometrical and physical characteristics of an elastic solid. Consequently, Lame's equations play a very prominent role in the solutions of problems.

Example 3.1 *A rubber cube is inserted in a cavity of the same form and size in a steel block and the top of the cube is pressed by a steel block with a pressure of p pascals. Considering the steel to be absolutely hard and assuming that there is no friction between steel and rubber, find (i) the pressure of rubber against the box walls, and (ii) the extremum shear stresses in rubber.*

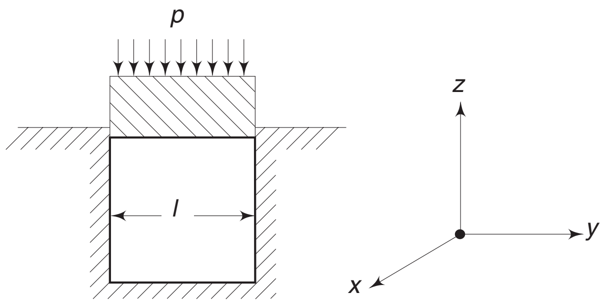

Fig. 3.1 *Example 3.1*

Solution

(i) Let l be the dimension of the cube. Since the cube is constrained in x and y directions

$$\varepsilon_{xx} = 0 \quad \text{and} \quad \varepsilon_{yy} = 0$$

and $\sigma_z = -p$

Therefore

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right] = 0$$

$$\varepsilon_{yy} = \frac{1}{E}\left[\sigma_y - \nu\left(\sigma_x + \sigma_z\right)\right] = 0$$

Solving

$$\sigma_x = \sigma_y = \frac{\nu}{1-\nu} \quad \sigma_z = -\frac{\nu}{1-\nu}\, p$$

If Poisson's ratio = 0.5, then

$$\sigma_x = \sigma_y = \sigma_z = -p$$

(ii) The extremum shear stresses are

$$\tau_2 = \frac{\sigma_1 - \sigma_3}{2}, \qquad \tau_3 = \frac{\sigma_1 - \sigma_2}{2}, \qquad \tau_1 = \frac{\sigma_2 - \sigma_3}{2}$$

If $\nu \le 0.5$, then σ_x and σ_y are numerically less than or equal to σ_z. Since σ_x, σ_y and σ_z are all compressive

$$\sigma_1 = \sigma_x = -\frac{\nu}{1-\nu}\, p$$

$$\sigma_2 = \sigma_y = -\frac{\nu}{1-\nu}\, p$$

$$\sigma_3 = \sigma_z = -p$$

$$\therefore \qquad \tau_1 = p\left(1 - \frac{\nu}{1-\nu}\right) = \frac{1-2\nu}{1-\nu}\, p, \quad \tau_2 = \frac{1-2\nu}{1-\nu}\, p, \quad \tau_3 = 0$$

If $\nu = 0.5$, the shear stresses are zero.

Example 3.2 *A cubical element is subjected to the following state of stress.*

$$\sigma_x = 100 \text{ MPa}, \quad \sigma_y = -20 \text{ MPa}, \quad \sigma_z = -40 \text{ Mpa}, \quad \tau_{xy} = \tau_{yz} = \tau_{zx} = 0$$

Assuming the material to be homogeneous and isotropic, determine the principal shear strains and the octahedral shear strain, if $E = 2 \times 10^5$ MPa and $\nu = 0.25$.

Solution Since the shear stresses on *x*, *y* and *z* planes are zero, the given stresses are principal stresses. Arranging such that $\sigma_1 \geq \sigma_2 \geq \sigma_3$

$$\sigma_1 = 100 \text{ MPa}, \quad \sigma_2 = -20 \text{ MPa}, \quad \sigma_3 = -40 \text{ MPa}$$

The extremal shear stresses are

$$\tau_1 = \frac{1}{2}(\sigma_2 - \sigma_3) = \frac{1}{2}(-20 + 40) = 10 \text{ Mpa}$$

$$\tau_2 = \frac{1}{2}(\sigma_3 - \sigma_1) = \frac{1}{2}(-40 - 100) = -70 \text{ Mpa}$$

$$\tau_3 = \frac{1}{2}(\sigma_1 - \sigma_2) = \frac{1}{2}(100 + 20) = 60 \text{ Mpa}$$

The modulus of rigidity G is

$$G = \frac{E}{2(1+\nu)} = \frac{2 \times 10^5}{2 \times 1.25} = 8 \times 10^4 \text{ MPa}$$

The principal shear strains are therefore

$$\gamma_1 = \frac{\tau_1}{G} = \frac{10}{8 \times 10^4} = 1.25 \times 10^{-4}$$

$$\gamma_2 = \frac{\tau_2}{G} = -\frac{70}{8 \times 10^4} = -8.75 \times 10^{-4}$$

$$\gamma_3 = \frac{\tau_3}{G} = \frac{60}{8 \times 10^4} = 7.5 \times 10^{-4}$$

From Eq. (1.44a), the octahedral shear stress is

$$\tau_0 = \frac{1}{3}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]^{1/2}$$

$$= \frac{1}{3}[120^2 + 20^2 + 140^2]^{1/2} = 61.8 \text{ MPa}$$

The octahedral shear strain is therefore

$$\gamma_0 = \frac{\tau_0}{G} = \frac{61.8}{8 \times 10^4} = 7.73 \times 10^{-4}$$

Problems

3.1 Compute Lame's coefficients λ and μ for
  (a) steel having $E = 207 \times 10^6$ kPa (2.1×10^6 kgf/cm^2) and $\nu = 0.3$.
  (b) concrete having $E = 28 \times 10^6$ kPa (2.85×10^5 kgf/cm^2) and $\nu = 0.2$.

[*Ans.* (a) 120×10^6 kPa (1.22×10^6 kgf/cm^2), 80×10^6 kPa (8.1680×10^5 kgf/cm^2)
(b) 7.8×10^6 kPa (7.96×10^4 kgf/cm^2), 11.7×10^6 kPa (1.2×10^5 kgf/cm^2)]

3.2 For steel, the following data is applicable:

$$E = 207 \times 10^6 \text{ kPa } (2.1 \times 10^6 \text{ kgf/cm}^2),$$

and $$G = 80 \times 10^6 \text{ kPa } (0.82 \times 10^6 \text{ kgf/cm}^2)$$

For the given strain matrix at a point, determine the stress matrix.

$$[\varepsilon_{ij}] = \begin{bmatrix} 0.001 & 0 & -0.002 \\ 0 & -0.003 & 0.0003 \\ -0.002 & 0.003 & 0 \end{bmatrix}$$

$$\left[Ans.\ [\tau_{ij}] = \begin{bmatrix} -68.4 & 0 & -160 \\ 0 & -708.4 & 24 \\ -160 & 24 & -228.4 \end{bmatrix} \times 10^3 \text{ kPa} \right]$$

3.3 A thin rubber sheet is enclosed between two fixed hard steel plates (see Fig. 3.2). Friction between the rubber and steel faces is negligible. If the rubber plate is subjected to stresses σ_x and σ_y as shown, determine the strains ε_{xx} and ε_{yy}, and also the stress ε_{zz}

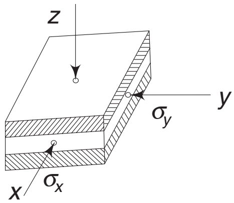

Fig. 3.2 *Example 3.2*

[*Ans.* $\sigma_z = +\nu(\sigma_x + \sigma_y)$

$$\varepsilon_{xx} = +\frac{1+\nu}{E}[(1-\nu)\sigma_x - \nu\sigma_y]$$

$$\varepsilon_{yy} = +\frac{1+\nu}{E}[(1-\nu)\sigma_y\ \nu\sigma_x]$$
]

Theories of Failure or Yield Criteria and Introduction to Ideally Plastic Solid

CHAPTER 4

4.1 INTRODUCTION

It is known from the results of material testing that when bars of ductile materials are subjected to uniform tension, the stress-strain curves show a linear range within which the materials behave in an elastic manner and a definite yield zone where the materials undergo permanent deformation. In the case of the so-called brittle materials, there is no yield zone. However, a brittle material, under suitable conditions, can be brought to a plastic state before fracture occurs. In general, the results of material testing reveal that the behaviour of various materials under similar test conditions, e.g. under simple tension, compression or torsion, varies considerably.

In the process of designing a machine element or a structural member, the designer has to take precautions to see that the member under consideration does not fail under service conditions. The word 'failure' used in this context may mean either fracture or permanent deformation beyond the operational range due to the yielding of the member. In Chapter 1, it was stated that the state of stress at any point can be characterised by the six rectangular stress components—three normal stresses and three shear stresses. Similarly, in Chapter 2, it was shown that the state of strain at a point can be characterised by the six rectangular strain components. When failure occurs, the question that arises is: what causes the failure? Is it a particular state of stress, or a particular state of strain or some other quantity associated with stress and strain? Further, the cause of failure of a ductile material need not be the same as that for a brittle material.

Consider, for example, a uniform rod made of a ductile material subject to tension. When yielding occurs,

(i) The principal stress σ at a point will have reached a definite value, usually denoted by σ_y;

(ii) The maximum shearing stress at the point will have reached a value equal to $\tau = \frac{1}{2}\sigma_y$;

(iii) The principal extension will have become $\varepsilon = \sigma_y/E$;

(iv) The octahedral shearing stress will have attained a value equal to $(\sqrt{2}/3)\,\sigma_y$;

and so on.

Any one of the above or some other factors might have caused the yielding. Further, as pointed out earlier, the factor that causes a ductile material to yield might be quite different from the factor that causes fracture in a brittle material under the same loading conditions. Consequently, there will be many criteria or theories of failure. It is necessary to remember that failure may mean fracture or yielding. Whatever may be the theory adopted, the information regarding it will have to be obtained from a simple test, like that of a uniaxial tension or a pure torsion test. This is so because the state of stress or strain which causes the failure of the material concerned can easily be calculated. The critical value obtained from this test will have to be applied for the stress or strain at a point in a general machine or a structural member so as not to initiate failure at that point.

There are six main theories of failure and these are discussed in the next section. Another theory, called Mohr's theory, is slightly different in its approach and will be discussed separately.

4.2 THEORIES OF FAILURE

Maximum Principal Stress Theory

This theory is generally associated with the name of Rankine. According to this theory, the maximum principal stress in the material determines failure regardless of what the other two principal stresses are, so long as they are algebraically smaller. This theory is not much supported by experimental results. Most solid materials can withstand very high hydrostatic pressures without fracture or without much permanent deformation if the pressure acts uniformly from all sides as is the case when a solid material is subjected to high fluid pressure. Materials with a loose or porous structure such as wood, however, undergo considerable permanent deformation when subjected to high hydrostatic pressures.On the other hand, metals and other crystalline solids (including consolidated natural rocks) which are impervious, are elastically compressed and can withstand very high hydrostatic pressures. In less compact solid materials, a marked evidence of failure has been observed when these solids are subjected to hydrostatic pressures. Further, it has been observed that even brittle materials, like glass bulbs, which are subject to high hydrostatic pressure do not fail when the pressure is acting, but fail either during the period the pressure is being reduced or later when the pressure is rapidly released. It is stated that the liquid could have penentrated through the fine invisible surface cracks and when the pressure was released, the entrapped liquid may not have been able to escape fast enough. Consequently, high pressure gradients are caused on the surface of the material which tend to burst or explode the glass. As Karman pointed out, this penentration and the consequent failure of the material can be prevented if the latter is covered by a thin flexible metal foil and then subjected to high hydrostatic pressures. Further noteworthy observations on the bursting action of a liquid which is used to transmit pressure were made by Bridgman who found that cylinders of hardened chrome-nickel steel were not able to withstand an internal pressure well if the liquid transmitting the pressure was mercury instead of viscous oil. It appears that small atoms of mercury are able to penentrate the cracks, whereas the large molecules of oil are not able to penentrate so easily.

From these observations, we draw the conclusion that a pure state of hydrostatic pressure [$\sigma_1 = \sigma_2 = \sigma_3 = -p$ ($p > 0$)] cannot produce permanent deformation in compact crystalline or amorphous solid materials but produces only a small elastic contraction, provided the liquid is prevented from entering the fine surface cracks or crevices of the solid. This contradicts the maximum principal stress theory. Further evidence to show that the maximum principal stress theory cannot be a good criterion for failure can be demonstrated in the following manner:

Consider the block shown in Fig. 4.1, subjected to stress σ_1 and σ_2, where σ_1 is tensile and σ_2 is compressive.

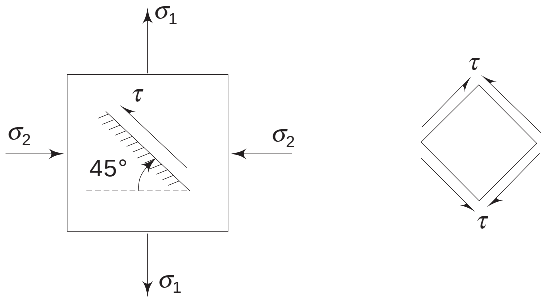

Fig. 4.1 *Rectangular element with 45° plane*

If σ_1 is equal to σ_2 in magnitude, then on a 45° plane, from Eq. (1.63b), the shearing stress will have a magnitude equal to σ_1. Such a state of stress occurs in a cylindrical bar subjected to pure torsion. If the maximum principal stress theory was valid, σ_1 would have been the limiting value. However, for ductile materials subjected to pure torsion, experiments reveal that the shear stress limit causing yield is much less than σ_1 in magnitude.

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2}$$

Notwithstanding all these, the maximum principal stress theory, because of its simplicity, is considered to be reasonably satisfactory for brittle materials which do not fail by yielding. Using information from a uniaxial tension (or compression) test, we say that failure occurs when the maximum principal stress at any point reaches a value equal to the tensile (or compressive) elastic limit or yield strength of the material obtained from the uniaxial test. Thus, if $\sigma_1 > \sigma_2 > \sigma_3$ are the principal stresses at a point and σ_y the yield stress or tensile elastic limit for the material under a uniaxial test, then failure occurs when

$$\sigma_1 \geq \sigma_y \tag{4.1}$$

Maximum Shearing Stress Theory

Observations made in the course of extrusion tests on the flow of soft metals through orifices lend support to the assumption that the plastic state in such metals is created when the maximum shearing stress just reaches the value of the resistance of the metal against shear. Assuming $\sigma_1 > \sigma_2 > \sigma_3$, yielding, according to this theory, occurs when the maximum shearing stress reaches a critical value. The maximum shearing stress theory is accepted to be fairly well justified for ductile materials. In a bar subject to uniaxial tension or compression, the maximum shear stress occurs on a plane at 45° to the load axis. Tension tests conducted on mild steel bars show that at the time of yielding, the so-called slip lines occur approximately at 45°, thus supporting the theory. On the other hand, for brittle crystalline materials which cannot be brought into the plastic state under tension but which may yield a little before fracture under compression, the angle of the slip planes or of the shear fracture surfaces, which usually develop along these planes, differs considerably from the planes of maximum shear. Further, in these brittle materials, the values of the maximum shear in tension and compression are not equal. Failure of material under triaxial tension (of equal magnitude) also does not support this theory, since equal triaxial tensions cannot produce any shear.

However, as remarked earlier, for ductile load carrying members where large shears occur and which are subject to unequal triaxial tensions, the maximum shearing stress theory is used because of its simplicity.

If $\sigma_1 > \sigma_2 > \sigma_3$ are the three principal stresses at a point, failure occurs when

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} \geq \frac{\sigma_y}{2} \tag{4.2}$$

where $\sigma_y/2$ is the shear stress at yield point in a uniaxial test.

Maximum Elastic Strain Theory

According to this theory, failure occurs at a point in a body when the maximum strain at that point exceeds the value of the maximum strain in a uniaxial test of the material at yield point. Thus, if σ_1, σ_2 and σ_3 are the principal stresses at a point, failure occurs when

$$\varepsilon_1 = \frac{1}{E}\left[\sigma_1 - \nu(\sigma_2 + \sigma_3)\right] \geq \frac{\sigma_y}{E} \tag{4.3}$$

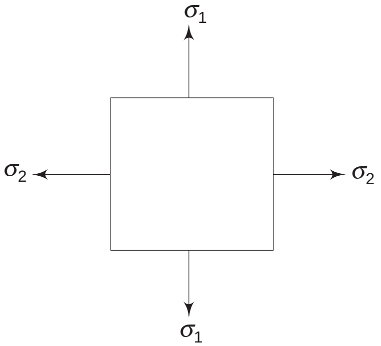

Fig. 4.2 *Biaxial state of stress*

We have observed that a material subjected to triaxial compression does not suffer failure, thus contradicting this theory. Also, in a block subjected to a biaxial tension, as shown in Fig. 4.2, the principal strain ε_1 is

$$\varepsilon_1 = \frac{1}{E}(\sigma_1 - \nu\sigma_2)$$

and is smaller than σ_1/E because of σ_2. Therefore, according to this theory, σ_1 can be increased more than σ_y without causing failure, whereas, if σ_2 were compressive, the magnitude of σ_1 to cause failure would be less than σ_y. However, this is not supported by experiments.

While the maximum strain theory is an improvement over the maximum stress theory, it is not a good theory for ductile materials. For materials which fail by brittle fracture, one may prefer the maximum strain theory to the maximum stress theory.

Octahedral Shearing Stress Theory

According to this theory, the critical quantity is the shearing stress on the octahedral plane. The plane which is equally inclined to all the three principal axes *Ox*, *Oy* and *Oz* is called the octahedral plane. The normal to this plane has direction cosines n_x, n_y and $n_z = 1/\sqrt{3}$. The tangential stress on this plane is the octahedral shearing stress. If σ_1, σ_2 and σ_3 are the principal stresses at a point, then from Eqs (1.44a) and (1.44c)

$$\tau_{\text{oct}} = \frac{1}{3}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2}$$

$$= \frac{\sqrt{2}}{3}\left(I_1^2 - 3I_2\right)^{1/2}$$

In a uniaxial test, at yield point, the octahedral stress $(\sqrt{2}/3)\ \sigma_y = 0.47\sigma_y$. Hence, according to the present theory, failure occurs at a point where the values of principal stresses are such that

$$\tau_{\text{oct}} = \frac{1}{3}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2} \geq \frac{\sqrt{2}}{3}\sigma_y \tag{4.4a}$$

or

$$\left(I_1^2 - 3I_2\right) \geq \sigma_y^2 \tag{4.4b}$$

This theory is supported quite well by experimental evidences. Further, when a material is subjected to hydrostatic pressure, $\sigma_1 = \sigma_2 = \sigma_3 = -p$, and τ_{oct} is equal to zero. Consequently, according to this theory, failure cannot occur and this, as stated earlier, is supported by experimental results. This theory is equivalent to the maximum distortion energy theory, which will be discussed subsequently.

Maximum Elastic Energy Theory

This theory is associated with the names of Beltrami and Haigh. According to this theory, failure at any point in a body subject to a state of stress begins only when the energy per unit volume absorbed at the point is equal to the energy absorbed per unit volume by the material when subjected to the elastic limit under a uniaxial state of stress. To calculate the energy absorbed per unit volume we proceed as follows:

Let σ_1, σ_2 and σ_3 be the principal stresses and let their magnitudes increase uniformly from zero to their final magnitudes. If ε_1, ε_2 and ε_3 are the corresponding principal strains, then the work done by the forces, from Fig. 4.3(b), is

$$\Delta W = \frac{1}{2}\sigma_1\,\Delta y\,\Delta z\,(\delta\Delta x) + \frac{1}{2}\sigma_2\,\Delta x\Delta z\,(\delta\Delta y) + \frac{1}{2}\sigma_3\,\Delta x\,\Delta y\,(\delta\Delta z)$$

where $\delta\Delta x$, $\delta\Delta y$ and $\delta\Delta z$ are extensions in *x*, *y* and *z* directions respectively.

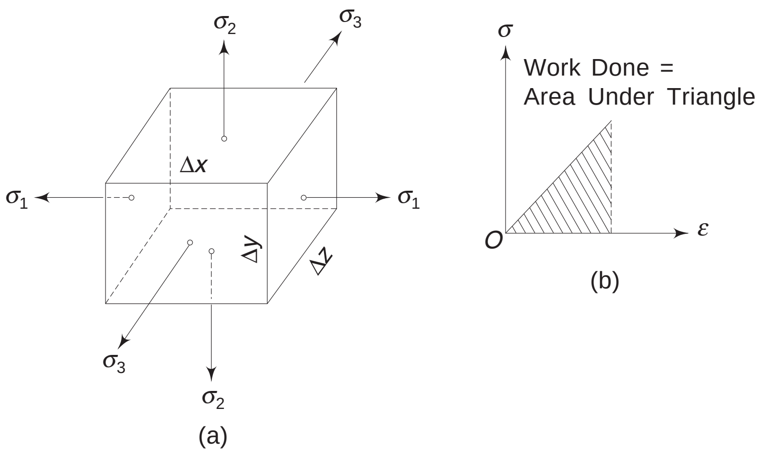

Fig. 4.3 *(a) Principal stresses on a rectangular block*
(b) Area representing work done

From Hooke' s law

$$\delta\Delta x = \varepsilon_1 \, \Delta x = \frac{1}{E}\left[\sigma_1 - \nu\left(\sigma_2 + \sigma_3\right)\right]\Delta x$$

$$\delta\Delta y = \varepsilon_2 \, \Delta y = \frac{1}{E}\left[\sigma_2 - \nu\left(\sigma_1 + \sigma_3\right)\right]\Delta y$$

$$\delta\Delta z = \varepsilon_3 \, \Delta z = \frac{1}{E}\left[\sigma_3 - \nu\left(\sigma_1 + \sigma_2\right)\right]\Delta z$$

Substituting these

$$\Delta W = \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\right)\right]\Delta x \, \Delta y \, \Delta z$$

The above work is stored as internal energy if the rate of deformation is small. Consequently, the energy U per unit volume is

$$\frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\right)\right] \tag{4.5}$$

In a uniaxial test, the energy stored per unit volume at yield point or elastic limit is $1/2E\,\sigma_y^2$. Hence, failure occurs when

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\right) \geq \sigma_y^2 \tag{4.6}$$

This theory does not have much significance since it is possible for a material to absorb considerable amount of energy without failure or permanent deformation when it is subjected to hydrostatic pressure.

Energy of Distortion Theory

This theory is based on the work of Huber, von Mises and Hencky. According to this theory, it is not the total energy which is the criterion for failure; in fact the energy absorbed during the distortion of an element is responsible for failure. The energy of distortion can be obtained by subtracting the energy of volumetric expansion from the total energy. It was shown in the Analysis of Stress (Sec. 1.22) that any given state of stress can be uniquely resolved into an isotropic state and a pure shear (or deviatoric) state. If σ_1, σ_2 and σ_3 are the principal stresses at a point then

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} = \begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & P \end{bmatrix} = \begin{bmatrix} \sigma_1 - p & 0 & 0 \\ 0 & \sigma_2 - p & 0 \\ 0 & 0 & \sigma_3 - p \end{bmatrix} \tag{4.7}$$

where $p = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$.

The first matrix on the right-hand side represents the isotropic state and the second matrix the pure shear state. Also, recall that the necessary and sufficient condition for a state to be a pure shear state is that its first invariant must be equal to zero. Similarly, in the Analysis of Strain (Section 2.17), it was shown that any given state of strain can be resolved uniquely into an isotropic and a deviatoric state of strain. If ε_1, ε_2 and ε_3 are the principal strains at the point, we have

$$\begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix} = \begin{bmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e \end{bmatrix} = \begin{bmatrix} \varepsilon_1 - e & 0 & 0 \\ 0 & \varepsilon_2 - e & 0 \\ 0 & 0 & \varepsilon_3 - e \end{bmatrix} \tag{4.8}$$

where $e = \frac{1}{3}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$.

It was also shown that the volumetric strain corresponding to the deviatoric state of strain is zero since its first invariant is zero.

It is easy to see from Eqs (4.7) and (4.8) that, by Hooke's law, the isotropic state of strain is related to the isotropic state of stress because

$$\varepsilon_1 = \frac{1}{E}\left[\sigma_1 - \nu\left(\sigma_2 + \sigma_3\right)\right]$$

$$\varepsilon_2 = \frac{1}{E}\left[\sigma_2 - \nu\left(\sigma_3 + \sigma_1\right)\right]$$

$$\varepsilon_3 = \frac{1}{E}\left[\sigma_3 - \nu\left(\sigma_2 + \sigma_1\right)\right]$$

Adding and taking the mean

$$\frac{1}{3}\left(\varepsilon_1 + \varepsilon_2 + \varepsilon_3\right) = e$$

$$= \frac{1}{3E}\left[\left(\sigma_1 + \sigma_2 + \sigma_3\right) - 2\nu\left(\sigma_1 + \sigma_2 + \sigma_3\right)\right]$$

or $$e = \frac{1}{E}\left[\left(1 - 2\nu\right) p\right] \tag{4.9}$$

i.e. e is connected to p by Hooke's law. This states that the volumetric strain $3e$ is proportional to the pressure p, the proportionality constant being equal to $\frac{3}{E}(1-2\nu) = K$, the bulk modulus, Eq. (3.14).

Consequently, the work done or the energy stored during volumetric change is

$$U' = \frac{1}{2}pe + \frac{1}{2}pe + \frac{1}{2}pe = \frac{3}{2}pe$$

Substituting for e from Eq. (4.9)

$$U' = \frac{3}{2E}(1-2\nu)\,p^2 \qquad (4.10)$$

$$= \frac{1-2\nu}{6E}(\sigma_1+\sigma_2+\sigma_3)^2$$

The total elastic strain energy density is given by Eq. (4.5). Hence, subtracting U' from U

$$U^* = \frac{1}{2E}\left(\sigma_1^2+\sigma_2^2+\sigma_3^2\right) - \frac{\nu}{E}\left(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1\right)$$

$$-\frac{1-2\nu}{6E}(\sigma_1+\sigma_2+\sigma_3)^2 \qquad (4.11a)$$

$$= \frac{2(1+\nu)}{6E}\left(\sigma_1^2+\sigma_2^2+\sigma_3^2-\sigma_1\sigma_2-\sigma_2\sigma_3-\sigma_3\sigma_1\right) \qquad (4.11b)$$

$$= \frac{(1+\nu)}{6E}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right] \qquad (4.11c)$$

Substituting $G = \dfrac{E}{2(1+\nu)}$ for the shear modulus,

$$U^* = \frac{1}{6G}\left(\sigma_1^2+\sigma_2^2+\sigma_3^2-\sigma_1\sigma_2-\sigma_2\sigma_3-\sigma_3\sigma_1\right) \qquad (4.12a)$$

or

$$U^* = \frac{1}{12G}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right] \qquad (4.12b)$$

This is the expression for the energy of distortion. In a uniaxial test, the energy of distortion is equal to $\frac{1}{6G}\sigma_y^2$. This is obtained by simply putting $\sigma_1 = \sigma_y$ and $\sigma_2 = \sigma_3 = 0$ in Eq. (4.12). This is also equal to $\frac{(1+\nu)}{3E}\sigma_y^2$ from Eq. (4.11c).

Hence, according to the distortion energy theory, failure occurs at that point where σ_1, σ_2 and σ_3 are such that

$$(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2 \ge 2\sigma_y^2 \qquad (4.13)$$

But we notice that the expression for the octahedral shearing stress from Eq. (1.22) is

$$\tau_{oct} = \frac{1}{3}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]^{1/2}$$

Hence, the distortion energy theory states that failure occurs when

$$9\tau_{oct}^2 = \geq 2\sigma_y^2$$

or

$$\tau_{oct} = \geq \frac{\sqrt{2}}{3}\sigma_y \tag{4.14}$$

This is identical to Eq. (4.4). Therefore, the octahedral shearing stress theory and the distortion energy theory are identical. Experiments made on the flow of ductile metals under biaxial states of stress have shown that Eq. (4.14) or equivalently, Eq. (4.13) expresses well the condition under which the ductile metals at normal temperatures start to yield. Further, as remarked earlier, the purely elastic deformation of a body under hydrostatic pressure ($\tau_{oct} = 0$) is also supported by this theory.

4.3 SIGNIFICANCE OF THE THEORIES OF FAILURE

The mode of failure of a member and the factor that is responsible for failure depend on a large number of factors such as the nature and properties of the material, type of loading, shape and temperature of the member, etc. We have observed, for example, that the mode of failure of a ductile material differs from that of a brittle material. While yielding or permanent deformation is the characteristic feature of ductile materials, fracture without permanent deformation is the characteristic feature of brittle materials. Further, if the loading conditions are suitably altered, a brittle material may be made to yield before failure. Even ductile materials fail in a different manner when subjected to repeated loadings (such as fatigue) than when subjected to static loadings. All these factors indicate that any rational procedure of design of a member requires the determination of the mode of failure (either yielding or fracture), and the factor (such as stress, strain and energy) associated with it. If tests could be performed on the actual member, subjecting it to all the possible conditions of loading that the member would be subjected to during operation, then one could determine the maximum loading condition that does not cause failure. But this may not be possible except in very simple cases. Consequently, in complex loading conditions, one has to identify the factor associated with the failure of a member and take precautions to see that this factor does not exceed the maximum allowable value. This information is obtained by performing a suitable test (uniform tension or torsion) on the material in the laboratory.

In discussing the various theories of failure, we have expressed the critical value associated with each theory in terms of the yield point stress σ_y obtained from a uniaxial tensile stress. This was done since it is easy to perform a uniaxial tensile stress and obtain the yield point stress value. It is equally easy to perform a pure torsion test on a round specimen and obtain the value of the maximum shear stress τ_y at the point of yielding. Consequently, one can also express the critical value associated with each theory of failure in terms of the yield point shear stress τ_y. In a sense, using σ_y or τ_y is equivalent because during a uniaxial tension, the maximum shear stress τ at a point is equal to $\frac{1}{2}\sigma$; and in the case of pure shear, the normal stresses on a 45° element are σ and $-\sigma$, where σ is numerically equivalent to τ. These are shown in Fig. 4.4.

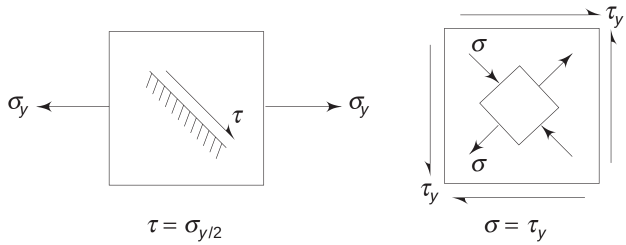

Fig. 4.4 *Uniaxial and pure shear state of stress*

If one uses the yield point shear stress τ_y obtained from a pure torsion test, then the critical value associated with each theory of failure is as follows:

(i) Maximum Normal Stress Theory According to this theory, failure occurs when the normal stress *s* at any point in the stressed member reaches a value

$$\sigma \geq \tau_y$$

This is because, in a pure torsion test when yielding occurs, the maximum normal stress *s* is numerically equivalent to t_y.

(ii) Maximum Shear Stress Theory According to this theory, failure occurs when the shear stress *t* at a point in the member reaches a value

$$\tau \geq \tau_y$$

(iii) Maximum Strain Theory According to this theory, failure occurs when the maximum strain at any point in the member reaches a value

$$\varepsilon = \frac{1}{E}[\sigma_1 - \nu(\sigma_2 + \sigma_3)]$$

From Fig. 4.4, in the case of pure shear

$$\sigma_1 = \sigma = \tau, \quad \sigma_2 = 0, \ \sigma_3 = -\sigma = -\tau$$

Hence, failure occurs when the strain *e* at any point in the member reaches a value

$$\varepsilon = \frac{1}{E}(\tau_y + \nu\tau_y) = \frac{1}{E}(1+\nu)\tau_y$$

(iv) Octahedral Shear Stress Theory When an element is subjected to pure shear, the maximum and minimum normal stresses at a point are *s* and –*s* (each numerically equal to the shear stress *t*), as shown in Fig. 4.4. Corresponding to this, from Eq. (1.44a), the octahedral shear stress is

$$\tau_{\text{oct}} = \frac{1}{3}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2}$$

Observing that $\sigma_1 = \sigma = \tau$, $\sigma_2 = 0$, $\sigma_3 = -\sigma = -\tau$

$$\tau_{\text{oct}} = \frac{1}{3}(\sigma^2 + \sigma^2 + 4\sigma^2)^{1/2}$$

$$= \frac{\sqrt{6}}{3}\sigma = \sqrt{\frac{2}{3}}\,\tau$$

So, failure occurs when the octahedral shear stress at any point is

$$\tau_{oct} = \sqrt{\frac{2}{3}}\,\tau_y$$

(v) Maximum Elastic Energy Theory The elastic energy per unit volume stored at a point in a stressed body is, from Eq. (4.5),

$$U = \frac{1}{E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\right)\right]$$

In the case of pure shear, from Fig. 4.4,

$$\sigma_1 = \tau, \qquad \sigma_2 = 0, \qquad \sigma_3 = -\tau$$

Hence,

$$U = \frac{1}{2E}\left[\tau^2 + \tau^2 - 2\nu\left(-\tau^2\right)\right]$$

$$= \frac{1}{E}(1+\nu)\,\tau^2$$

So, failure occurs when the elastic energy density at any point in a stressed body is such that

$$U = \frac{1}{E}(1+\nu)\,\tau_y^2$$

(vi) Distortion Energy Theory The distortion energy density at a point in a stressed body is, from Eq. (4.12),

$$U^* = \frac{1}{12G}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]$$

Once again, by observing that in the case of pure shear

$$\sigma_1 = \tau, \qquad \sigma_2 = 0, \qquad \sigma_3 = -\tau$$

$$U^* = \frac{1}{12G}\left[\tau^2 + \tau^2 + 4\tau^2\right]$$

$$= \frac{1}{2G}\,\tau^2$$

So, failure occurs when the distortion energy density at any point is equal to

$$U^* = \frac{1}{2G}\,\tau_y^2 = \frac{1}{2}\cdot\frac{2(1+\nu)}{E}\,\tau_y^2$$

$$= \frac{(1+\nu)}{E}\,\tau_y^2$$

The foregoing results show that one can express the critical value associated with each theory of failure either in terms of σ_y or in terms of τ_y. Assuming that a particular theory of failure is correct for a given material, then the values of σ_y and τ_y obtained from tests conducted on the material should be related by the corresponding expressions. For example, if the distortion energy is a valid theory for a material, then the value of the energy in terms of σ_y and that in terms of τ_y should be equal. Thus,

$$U^* = \frac{(1+\nu)}{E}\tau_y^2 = \frac{(1+\nu)}{3E}\sigma_y^2$$

or

$$\tau_y = \frac{1}{\sqrt{3}}\sigma_y = 0.577\,\sigma_y$$

This means that the value of τ_y obtained from pure torsion test should be equal to 0.577 times the value of σ_y obtained from a uniaxial tension test conducted on the same material.

Table 4.1 summarizes these theories and the corresponding expressions. The first column lists the six theories of failure. The second column lists the critical value associated with each theory in terms of σ_y, the yield point stress in uniaxial tension test. For example, according to the octahedral shear stress theory, failure occurs when the octahedral shear stress at a point assumes a value equal to $\sqrt{2}/3\,\sigma_y$. The third column lists the critical value associated with each theory in terms of τ_y, the yield point shear stress value in pure torsion. For example, according to octahedral shear stress theory, failure occurs at a point when the octahedral shear stress equals a value $\sqrt{2/3}\,\tau_y$. The fourth column gives the relationship that should exist between τ_y and σ_y in each case if each theory is valid. Assuming octahedral shear stress theory is correct, then the value of τ_y obtained from pure torsion test should be equal to 0.577 times the yield point stress σ_y obtained from a uniaxial tension test.

Tests conducted on many ductile materials reveal that the values of τ_y lie between 0.50 and 0.60 of the tensile yield strength σ_y, the average value being about 0.57. This result agrees well with the octahedral shear stress theory and the

Table 4.1

Failure theory	*Tension*	*Shear*	*Relationship*
Max. normal stress	σ_y	$\sigma_y = \tau_y$	$\tau_y = \sigma_y$
Max. shear stress	$\tau = \frac{1}{2}\sigma_y$	τ_y	$\tau_y = 0.5\,\sigma_y$
Max. strain $\left(\nu = \frac{1}{4}\right)$	$\varepsilon = \frac{1}{E}\sigma_y$	$\varepsilon = \frac{5}{4}\frac{\tau_y}{E}$	$\tau_y = 0.8\,\sigma_y$
Octahedral shear	$\tau_{oct} = \frac{\sqrt{2}}{3}\sigma_y$	$\tau_{oct} = \sqrt{\frac{2}{3}}\,\tau_y$	$\tau_y = 0.577\,\sigma_y$
Max. energy $\left(\nu = \frac{1}{4}\right)$,	$U = \frac{1}{2E}\sigma_y^2$	$U = \frac{5}{4}\frac{1}{E}\tau_y^2$	$\tau_y = 0.632\,\sigma_y$
Distortion energy	$U^* = \frac{1+\nu}{3}\frac{\sigma_y^2}{E}$	$U^* = (1+\nu)\frac{\tau_y^2}{E}$	$\tau_y = 0.577\,\sigma_y$

distortion energy theory. The maximum shear stress theory predicts that shear yield value τ_y is 0.5 times the tensile yield value. This is about 15% less than the value predicted by the distortion energy (or the octahedral shear) theory. The maximum shear stress theory gives values for design on the safe side. Also, because of its simplicity, this theory is widely used in machine design dealing with ductile materials.

4.4 USE OF FACTOR OF SAFETY IN DESIGN

In designing a member to carry a given load without failure, usually a factor of safety N is used. The purpose is to design the member in such a way that it can carry N times the actual working load without failure. It has been observed that one can associate different factors for failure according to the particular theory of failure adopted. Consequently, one can use a factor appropriately reduced during the design process. Let X be a factor associated with failure and let F be the load. If X is directly proportional to F, then designing the member to safely carry a load equal to NF is equivalent to designing the member for a critical factor equal to X/N. However, if X is not directly proportional to F, but is, say, proportional to F^2, then designing the member to safely carry a load to equal to NF is equivalent to limiting the critical factor to $\sqrt{X/N}$. Hence, in using the factor of safety, care must be taken to see that the critical factor associated with failure is not reduced by N, but rather the load-carrying capacity is increased by N. This point will be made clear in the following example.

Example 4.1 *Determine the diameter d of a circular shaft subjected to a bending moment M and a torque T, according to the several theories of failure. Use a factor of safety N.*

Solution Consider a point P on the periphery of the shaft. If d is the diameter, then owing to the bending moment M, the normal stress σ at P on a plane normal to the axis of the shaft is, from elementary strength of materials,

$$\sigma = \frac{My}{I} = M\frac{d}{2}\frac{64}{\pi d^4} \tag{4.15}$$

$$= \frac{32M}{\pi d^3}$$

The shearing stress on a transverse plane at P due to torsion T is

$$\tau = \frac{Td}{2I_P} = \frac{Td \cdot 32}{2\pi d^4} \tag{4.16}$$

$$= \frac{16T}{\pi d^3}$$

Therefore, the principal stresses at P are

$$\sigma_{1,3} = \frac{1}{2}\sigma \pm \frac{1}{2}\sqrt{\left(\sigma^2 + 4\tau^2\right)}, \quad \sigma_2 = 0 \tag{4.17}$$

(i) Maximum Normal Stress Theory At point P, the maximum normal stress should not exceed s_y, the yield point stress in tension. With a factor of safety N, when the load is increased N times, the normal and shearing stresses are Ns and Nt. Equating the maximum normal stress to s_y,

$$\sigma_{\max} = \sigma_1 = N\left[\frac{\sigma}{2} + \frac{1}{2}\left(\sigma^2 + 4\tau^2\right)^{1/2}\right] = \sigma_y$$

or
$$\sigma + \left(\sigma^2 + 4\tau^2\right)^{1/2} = \frac{2\sigma_y}{N}$$

i.e.,
$$\frac{32M}{\pi d^3} + \frac{1}{\pi d^3} \times 32\left(M^2 + T^2\right)^{1/2} = \frac{2\,\sigma_y}{N}$$

i.e.,
$$16M + 16\left(M^2 + T^2\right)^{1/2} = \frac{\pi d^3 \sigma_y}{N}$$

From this, the value of d can be determined with the known values of M, T and s_y.

(ii) Maximum Shear Stress Theory At point P, the maximum shearing stress from Eq. (4.17) is

$$\tau_{\max} = \frac{1}{2}\left(\sigma_1 - \sigma_3\right) = \frac{1}{2}\left(\sigma^2 + 4\tau^2\right)^{1/2}$$

When the load is increased N times, the shear stress becomes Nt.
Hence,

$$N\tau_{\max} = \frac{1}{2} N\left(\sigma^2 + 4\tau^2\right)^{1/2} = \frac{\sigma_y}{2}$$

or,
$$\left(\sigma^2 + 4\tau^2\right)^{1/2} = \frac{\sigma_y}{N}$$

Substituting for σ and τ

$$\frac{32}{\pi d^3}\left(M^2 + T^2\right)^{1/2} = \frac{\sigma_y}{N}$$

or,
$$32\left(M^2 + T^2\right)^{1/2} = \frac{\pi d^3 \sigma_y}{N}$$

(iii) Maximum Strain Theory The maximum elastic strain at point P with a factor of safety N is

$$\varepsilon_{\max} = \frac{N}{E}\left[\sigma_1 - \nu\left(\sigma_2 + \sigma_3\right)\right]$$

From Eq. (4.3)

$$\sigma_1 - \nu\left(\sigma_2 + \sigma_3\right) = \frac{\sigma_y}{N}$$

Since $\sigma_2 = 0$, we have $\sigma_1 - \nu\,\sigma_3 = \dfrac{\sigma_y}{N}$

or $$\frac{\sigma}{2}+\frac{1}{2}\left(\sigma^2+4\tau^2\right)^{1/2}-\nu\,\frac{\sigma}{2}+\frac{\nu}{2}\left(\sigma^2+4\tau^2\right)^{1/2}=\frac{\sigma_y}{N}$$

Substituting for σ and τ

$$(1-\nu)\frac{16M}{\pi d^3}+(1+\nu)\frac{16}{\pi d^3}\left(M^2+T^2\right)^{1/2}=\frac{\sigma_y}{N}$$

or $$(1-\nu)16M+(1+\nu)16\left(M^2+T^2\right)^{1/2}=\frac{\pi d^3\sigma_y}{N}$$

(iv) Octahedral Shear Stress Theory The octahedral shearing stress at point P from Eq. (4.4a), and using a factor of safety N, is

$$N\tau_{oct}=\frac{N}{3}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]^{1/2}=\frac{\sqrt{2}}{3}\sigma_y$$

or $$\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]^{1/2}=\frac{\sqrt{2}}{N}\sigma_y$$

With $\sigma_2 = 0$

$$\left[2\sigma_1^2+2\sigma_3^2-2\sigma_1\,\sigma_3\right]^{1/2}=\frac{\sqrt{2}}{N}\sigma_y$$

or $$\left[\sigma_1^2+\sigma_3^2-\sigma_1\,\sigma_3\right]^{1/2}=\frac{\sigma_y}{N}$$

Substituting for σ_1 and σ_3

$$\Bigg[\frac{1}{4}\sigma^2+\frac{1}{4}\left(\sigma^2+4\tau^2\right)+\frac{1}{2}\sigma\left(\sigma^2+4\tau^2\right)^{1/2}+\frac{1}{4}\sigma^2+\frac{1}{4}\left(\sigma^2+4\tau^2\right)$$

$$-\frac{1}{2}\sigma\left(\sigma^2+4\tau^2\right)^{1/2}-\frac{1}{4}\sigma^2+\frac{1}{4}\left(\sigma^2+4\tau^2\right)\Bigg]^{1/2}=\frac{\sigma_y}{N}$$

or $$\left(\sigma^2+3\tau^2\right)^{1/2}=\frac{\sigma_y}{N}$$

Substituting for σ and τ

$$\frac{16}{\pi d^3}\left(4M^2+3T^2\right)^{1/2}=\frac{\sigma_y}{N}$$

or $$16\,(4M^2+3T^2)^{1/2}=\frac{\pi d^3\,\sigma_y}{N}$$

(v) Maximum Energy Theory The maximum elastic energy at P from Eq. (4.6) and with a factor of safety N is

$$U=\frac{N^2}{2E}\left[\sigma_1^2+\sigma_2^2+\sigma_3^2-2\nu\left(\sigma_1\,\sigma_2+\sigma_2\,\sigma_3+\sigma_3\,\sigma_1\right)\right]=\frac{\sigma_y^2}{2E}$$

Note: Since the stresses for design are $N\sigma_1$, $N\sigma_2$ and $N\sigma_3$, the factor N^2 appears in the expression for U. In the previous four cases, only N appeared because of the particular form of the expression.

With $\sigma_2 = 0$,

$$(\sigma_1^2 + \sigma_3^2 - 2\nu\,\sigma_1\,\sigma_3) = \frac{\sigma_y^2}{N^2}$$

Substituting for σ_1 and σ_3

$$\left[\frac{1}{4}\sigma^2 + \frac{1}{4}\left(\sigma^2 + 4\tau^2\right) + \frac{1}{2}\sigma\left(\sigma^2 + 4\tau^2\right)^{1/2} + \frac{1}{4}\sigma^2\right.$$

$$\left. + \frac{1}{4}\left(\sigma^2 + 4\tau^2\right) - \frac{1}{2}\sigma\left(\sigma^2 + 4\tau^2\right)^{1/2} + 2\nu\tau^2\right] = \frac{\sigma_y^2}{N^2}$$

or $$\sigma^2 + (2 + 2\nu)\,\tau^2 = \frac{\sigma_y^2}{N^2}$$

i.e. $$\left[\sigma^2 + (2 + 2\nu)\,\tau^2\right]^{1/2} = \frac{\sigma_y}{N}$$

i.e. $$\frac{16}{\pi d^3}\left[4M^2 + (2 + 2\nu)\,T^2\right]^{1/2} = \frac{\sigma_y}{N}$$

or $$\left[4M^2 + 2(1 + \nu)\,T^2\right]^{1/2} = \frac{\pi d^3 \sigma_y}{16\,N}$$

(vi) Maximum Distortion Energy Theory The distortion energy associated with Ns_1, Ns_2 and Ns_3 at P is given by Eq. (4.11c). Equating this to distortion energy in terms of s_y

$$U_d = \frac{N^2(1+\nu)}{6E}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]$$

$$= \frac{1+\nu}{3E}\,\sigma_y^2$$

With $\sigma_2 = 0$,

$$(2\sigma_1^2 + 2\sigma_3^2 - 2\sigma_1\,\sigma_3) = \frac{2\,\sigma_y^2}{N^2}$$

or $$(\sigma_1^2 + \sigma_3^2 - \sigma_1\,\sigma_3)^{1/2} = \frac{\sigma_y}{N}$$

This yields the same result as the octahedral shear stress theory.

4.5 A NOTE ON THE USE OF FACTOR OF SAFETY

As remarked earlier, when a factor of safety N is prescribed, we may consider two ways of introducing it in design:

(i) Design the member so that it safely carries a load NF.

(ii) If the factor associated with failure is X, then see that this factor at any point in the member does not exceed X/N.

But the second method of using N is not correct, since by the definition of the factor of safety, the member is to be designed for N times the load. So long as X is directly proportional to F, whether one uses NF or X/N for design analysis, the result will be identical. If X is not directly proportional to F, method (ii) may give wrong results. For example, if we adopt method (ii) with the maximum energy theory, the result will be

$$U = \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu\left(\sigma_1\,\sigma_2 + \sigma_2\,\sigma_3 + \sigma_3\,\sigma_1\right)\right] = \frac{1}{N}\,\frac{\sigma_y^2}{2E}$$

where X, the factor associated with failure, is $\frac{1}{2}\frac{\sigma_y^2}{E}$. But method (i) gives

$$U = \frac{N^2}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu\left(\sigma_1\,\sigma_2 + \sigma_2\,\sigma_3 + \sigma_3\,\sigma_1\right)\right] = \frac{\sigma_y^2}{2E}$$

The result obtained from method (i) is correct, since $N\sigma_1$, $N\sigma_2$ and $N\sigma_3$ are the principal stresses corresponding to the load NF. As one an see, the results are not the same. The result given by method (ii) is not the right one.

Example 4.2 *A force F = 45,000 N is necessary to rotate the shaft shown in Fig. 4.5 at uniform speed. The crank shaft is made of ductile steel whose elastic limit is 207,000 kPa, both in tension and compression. With $E = 207 \times 10^6$ kPa, ν = 0.25, determine the diameter of the shaft, using the octahedral shear stress theory and the maximum shear stress theory. Use a factor of safety N = 2. Consider a point on the periphery at section A for analysis.*

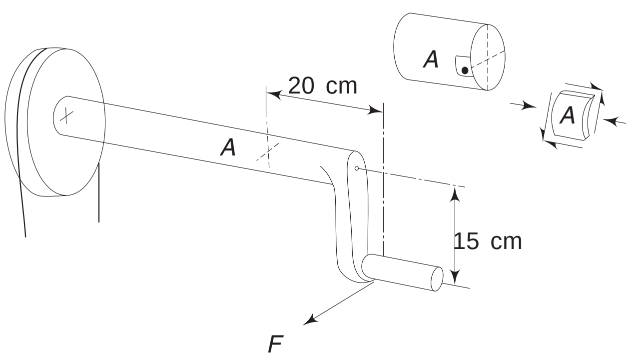

Fig. 4.5 *Example 4.2*

Solution The moment at section A is

$$M = 45{,}000 \times 0.2 = 9000 \text{ Nm}$$

and the torque on the shaft is

$$T = 45{,}000 \times 0.15 = 6750 \text{ Nm}$$

The normal stress due to M at A is

$$\sigma = -\frac{64Md}{2\pi d^4} = -\frac{32M}{\pi d^3}$$

and the maximum shear stress due to T at A is

$$\tau = \frac{32Td}{2\pi d^4} = \frac{16T}{\pi d^3}$$

The shear stress due to the shear force F is zero at A. The principal stresses from Eq. (1.61) are

$$\sigma_{1,3} = \frac{1}{2}\sigma \pm \frac{1}{2}\left(\sigma^2 + 4\tau^2\right)^{1/2}, \qquad \sigma_2 = 0$$

(i) ***Maximum Shear Stress Theory***

$$\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_3)$$

$$= \frac{1}{2}(\sigma^2 + 4\tau^2)^{1/2}$$

$$= \frac{1}{2}\frac{32}{\pi d^3}(M^2 + T^2)^{1/2}$$

$$= \frac{16}{\pi d^3}\left(9000^2 + 6750^2\right)^{1/2} = \frac{57295.8}{d^3}\ \text{Pa}$$

With a factor of safety $N = 2$, the value of τ_{max} becomes

$$N\tau_{max} = \frac{114591.6}{d^3}\ \text{Pa}$$

This should not exceed the maximum shear stress value at yielding in uniaxial tension test. Thus,

$$\frac{1}{d^3}(114591.6) = \frac{\sigma_y}{2} = \frac{207}{2} \times 10^6$$

$$\therefore \qquad d^3 = 1107 \times 10^{-6}\ \text{m}^3$$

$$\text{or} \qquad d = 10.35 \times 10^{-2}\ \text{m} = 10.4\ \text{cm}$$

(ii) Octahedral Shear Stress Theory

$$\tau_{oct} = \frac{1}{3}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2}$$

With $\sigma_2 = 0$,

$$\tau_{oct} = \frac{1}{3}\left[2\sigma_1^2 + 2\sigma_3^2 - 2\sigma_1\sigma_3\right]^{1/2}$$

Substituting for σ_1 and σ_3 and simplifying

$$\tau_{oct} = \frac{\sqrt{2}}{3}\left(\sigma^2 + 3\tau^2\right)^{1/2}$$

$$= \frac{\sqrt{2}}{3\pi d^3}\left[(32M)^2 + 3(16T)^2\right]^{1/2}$$

$$= \frac{16\sqrt{2}}{3\pi d^3}\left(4M^2 + 3T^2\right)^{1/2}$$

$$= \frac{16\sqrt{2}}{3\pi d^3}\left[4(9000)^2 + 3(6750)^2\right]^{1/2}$$

$$= \frac{\sqrt{2}}{3\pi d^3} \times 343418$$

Equating this to octahedral shear stress at yielding of a uniaxial tension bar, and using a factor of safety $N = 2$,

$$\frac{\sqrt{2}}{3\pi d^3} \times 2 \times 343418 = \frac{\sqrt{2}}{3}\sigma_y$$

or $\quad 2 \times 343418 = \pi d^3 \sigma_y = \pi d^3 \times 207 \times 10^6$

$\therefore \quad d^3 = 1.056 \times 10^{-3}$

or $\quad d = 0.1018 \text{ m} = 10.18 \text{ cm}$

Example 4.3 *A cylindrical bar of 7 cm diameter is subjected to a torque equal to 3400 Nm, and a bending moment M. If the bar is at the point of failing in accordance with the maximum principal stress theory, determine the maximum bending moment it can support in addition to the torque. The tensile elastic limit for the material is 207 MPa, and the factor of safety to be used is 3.*

Solution From Example 4.1(i)

$$16M + 16(M^2 + T^2)^{1/2} = \frac{\pi d^3}{N}\sigma_y$$

i.e. $\quad 16M + 16(M^2 + 3400^2)^{1/2} = \dfrac{\pi \times 7^3 \times 10^{-6} \times 207 \times 10^6}{3}$

or $\quad (M^2 + 3400^2)^{1/2} = 4647 - M$

or $\quad M^2 + 3400^2 = 4647^2 + M^2 - 9294M$

$\therefore \quad M = 1080 \text{ Nm}$

Example 4.4 *In Example 4.3, if failure is governed by the maximum strain theory, determine the diameter of the bar if it is subjected to a torque T = 3400 Nm and a bending moment M = 1080 Nm. The elastic modulus for the material is* $E = 103 \times 10^6$ *kPa,* $\nu = 0.25$*, factor of safety N = 3 and* $\sigma_y = 207$ *MPa.*

Solution According to the maximum strain theory and Example 4.1(iii)

$$16(1-\nu)M + 16(1+\nu)(M^2 + T^2)^{1/2} = \frac{\pi d^3}{N}\sigma_y$$

$$(16 \times 0.75 \times 1080) + (16 \times 1.25)(1080^2 + 3400^2)^{1/2} = \frac{\pi d^3}{3} \times 207 \times 10^6$$

i.e., $12960 + 71348 = 216.77 \times 10^6\, d^3$

or $d^3 = 389 \times 10^{-6}$

or $d = 7.3 \times 10^{-2}\,\text{m} = 7.3\ \text{cm}$

Example 4.5 *An equipment used in deep sea investigation is immersed at a depth H. The weight of the equipment in water is W. The rope attached to the instrument has a specific weight γ_r and the water has a specific weight γ. Analyse the strength of the rope. The rope has a cross-sectional area A. (Refer to Fig. 4.6.)*

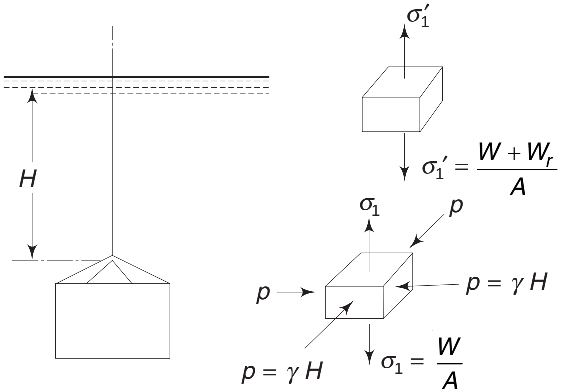

Fig. 4.6 *Example 4.5*

Solution The lower end of the rope is subjected to a triaxial state of stress. There is a tensile stress σ_1 due to the weight of the equipment and two hydrostatic compressions each equal to p, where

$$\sigma_1 = \frac{W}{A}, \qquad \sigma_2 = \sigma_3 = -\gamma H \quad \text{(compression)}$$

At the upper section there is only a uniaxial tension σ_1' due to the weight of the equipment and rope immersed in water.

$$\sigma_1' = \frac{W}{A} + (\gamma_r - \gamma)\, H; \qquad \sigma_2' = \sigma_3' = 0$$

Therefore, according to the maximum shear stress theory, at lower section

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{1}{2}\left(\frac{W}{A} + \gamma H\right)$$

and at the upper section

$$\tau_{\max} = \frac{\sigma_1' - \sigma_3'}{2} = \frac{1}{2}\left(\frac{W}{A} - \gamma H + \gamma_r H\right)$$

If the specific weight of the rope is more than twice that of water, then the upper section is the critical section. When the equipment is above the surface of the water, near the hoist, the stress is

$$\sigma_1 = \frac{W'}{A} \quad \text{and} \quad \sigma_2 = \sigma_3 = 0$$

$$\tau_{max} = \frac{1}{2}\frac{W'}{A}$$

W′ is the weight of equipment in air and is more than *W*. It is also necessary to check the strength of the rope for this stress.

4.6 MOHR'S THEORY OF FAILURE

In the previous discussions on failure, all the theories had one common feature. This was that the criterion of failure is unaltered by a reversal of sign of the stress. While the yield point stress σ_y for a ductile material is more or less the same in tension and compression, this is not true for a brittle material. In such a case, according to the maximum shear stress theory, we would get two different values for the critical shear stress. Mohr's theory is an attempt to extend the maximum shear stress theory (also known as the stress-difference theory) so as to avoid this objection.

To explain the basis of Mohr's theory, consider Mohr's circles, shown in Fig. 4.7, for a general state of stress.

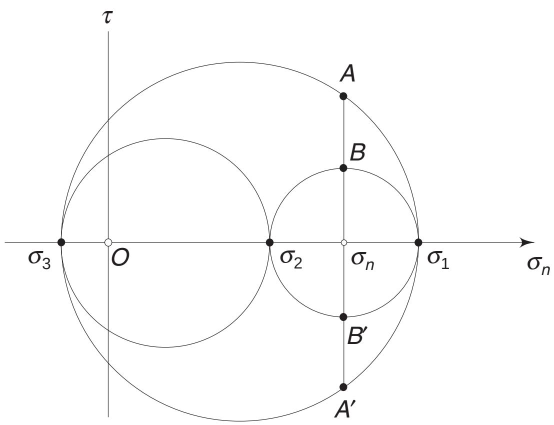

Fig. 4.7 *Mohr's circles*

σ_1, σ_2 and σ_3 are the principal stresses at the point. Consider the line *ABB′A′*. The points lying on *BA* and *B′A′* represent a series of planes on which the normal stresses have the same magnitude σ_n but different shear stresses. The maximum shear stress associated with this normal stress value is τ, represented by point *A* or *A′*. The fundamental assumption is that if failure is associated with a given normal stress value, then the plane having this normal stress and a maximum shear stress accompanying it, will be the critical plane. Hence, the critical point for the normal stress σ_n will be the point *A*. From Mohr's circle diagram, the planes having maximum shear stresses for given normal stresses, have their representative points on the outer circle. Consequently, as far as failure is concerned, the critical circle is the outermost circle in Mohr's circle diagram, with diameter $(\sigma_1 - \sigma_3)$.

Now, on a given material, we conduct three experiments in the laboratory, relating to simple tension, pure shear and simple compression. In each case, the test is conducted until failure occurs. In simple tension, $\sigma_1 = \sigma_{yt}$, $\sigma_2 = \sigma_3 = 0$. The outermost circle in the circle diagram (there is only one circle) corresponding to this state is shown as T in Fig. 4.8. The plane on which failure occurs will have its representative point on this outer circle. For pure shear, $\tau_{ys} = \sigma_1 = -\sigma_3$ and $\sigma_2 = 0$. The outermost circle for this state is indicated by S. In simple compression, $\sigma_1 = \sigma_2 = 0$ and $\sigma_3 = -\sigma_{yc}$. In general, for a brittle material, σ_{yc} will be greater than σ_{yt} numerically. The outermost circle in the circle diagram for this case is represented by C.

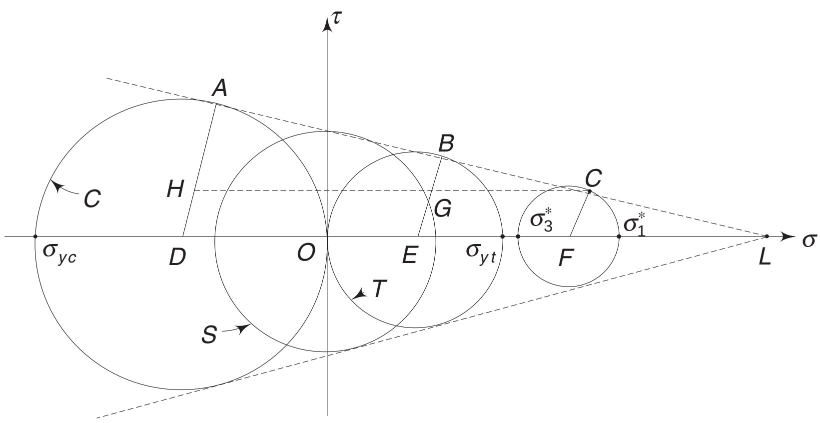

Fig. 4.8 *Diagram representing Mohr's failure theory*

In addition to the three simple tests, we can perform many more tests (like combined tension and torsion) until failure occurs in each case, and correspondingly for each state of stress, we can construct the outermost circle. For all these circles, we can draw an envelope. The point of contact of the outermost circle for a given state with this envelope determines the combination of σ and τ, causing failure. Obviously, a large number of tests will have to be performed on a single material to determine the envelope for it.

If the yield point stress in simple tension is small, compared to the yield point stress in simple compression, as shown in Fig. 4.8, then the envelope will cut the horizontal axis at point L, representing a finite limit for 'hydrostatic tension'. Similarly, on the left-hand side, the envelope rises indefinitely, indicating no elastic limit under hydrostatic compression.

For practical application of this theory, one assumes the envelopes to be straight lines, i.e. tangents to the circles as shown in Fig. 4.8. When a member is subjected to a general state of stress, for no failure to take place, the Mohr's circle with $(\sigma_1 - \sigma_3)$ as diameter should lie within the envelope. In the limit, the circle can touch the envelope. If one uses a factor of safety N, then the circle with $N(\sigma_1 - \sigma_3)$ as diameter can touch the envelopes. Figure 4.8 shows this limiting state of stress, where $\sigma_1^* = N\sigma_1$ and $\sigma_3^* = N\sigma_3$.

The envelopes being common tangents to the circles, triangles *LCF*, *LBE* and *LAD* are similar. Draw *CH* parallel to *LO* (the σ axis), making *CBG* and *CAH* similar. Then,

$$\frac{BG}{CG} = \frac{AH}{CH} \tag{a}$$

Now, $$BG = BE - GE = BE - CF = \frac{1}{2}\sigma_{yt} - \frac{1}{2}(\sigma_1^* - \sigma_3^*)$$

$$CG = FE = FO - EO = \frac{1}{2}(\sigma_1^* + \sigma_3^*) - \frac{1}{2}\sigma_{yt}$$

$$AH = AD - HD = AD - CF = \frac{1}{2}\sigma_{yc} - \frac{1}{2}(\sigma_1^* - \sigma_3^*)$$

$$CH = FD = FO + OD = \frac{1}{2}(\sigma_1^* + \sigma_3^*) + \frac{1}{2}\sigma_{yc}$$

Substituting these in Eq. (a), and after simplification,

$$\sigma_{yt} = \sigma_1^* - \frac{\sigma_{yt}}{\sigma_{yc}}\sigma_3^*$$

$$= N(\sigma_1 - k\sigma_3) \tag{4.18a}$$

where $$k = \frac{\sigma_{yt}}{\sigma_{yc}} \tag{4.18b}$$

Equation (4.18a) states that for a general state of stress where σ_1 and σ_3 are the maximum and minimum principal stresses, to avoid failure according to Mohr's theory, the condition is

$$\sigma_1 - k\sigma_3 \leq \frac{\sigma_{yt}}{N} = \sigma_{eq}$$

where *N* is the factor of safety used for design, and *k* is the ratio of σ_{yt} to σ_{yc} for the material. For a brittle material with no yield stress value, *k* is the ratio of σ ultimate in tension to σ ultimate in compression, i.e.

$$k = \frac{\sigma_{ut}}{\sigma_{uc}} \tag{4.18c}$$

σ_{yt}/N is sometimes called the equivalent stress σ_{eq} in uniaxial tension corresponding to Mohr's theory of failure. When $\sigma_{yt} = \sigma_{yc}$, *k* will become equal to 1 and Eq. (4.18a) becomes identical to the maximum shear stress theory, Eq. (4.2).

Example 4.6 *Consider the problem discussed in Example 4.2. Let the crank-shaft material have $\sigma_{yt} = 150$ MPa and $\sigma_{yc} = 330$ MPa. If the diameter of the shaft is 10 cm, determine the allowable force F according to Mohr's theory of failure. Let the factor of safety be 2. Consider a point on the surface of the shaft where the stress due to bending is maximum.*

Solution Bending moment at section $A = (20 \times 10^{-2}\ F)$ Nm

$$\text{Torque} = (15 \times 10^{-2}\ F)\ \text{Nm}$$

$$\therefore \qquad \sigma\,(\text{bending}) = \frac{64Md}{2\pi d^4} = \frac{32M}{\pi d^3}\ \text{Pa}$$

$$\tau\,(\text{torsion}) = \frac{32Td}{2\pi d^4} = \frac{16T}{\pi d^3}\ \text{Pa}$$

$$\sigma_{1,3} = \frac{1}{2}\sigma \pm \frac{1}{2}(\sigma^2 + 4\tau^2)^{1/2}, \qquad \sigma_2 = 0$$

$$\sigma_{1,3} = \frac{16M}{\pi d^3} \pm \frac{8}{\pi d^4}(4M^2 + T^2)^{1/2}$$

$$= \frac{8F}{\pi \times 10^{-3}}\left[2\,(20\times 10^{-2}) \pm 10^{-2}\,(1600 + 22F)^{1/2}\right]$$

$$= \frac{80F}{\pi}(40 \pm 42.7) = 2106F; \qquad -68.75F$$

$$k = \frac{\sigma_{yt}}{\sigma_{yc}} = \frac{150}{330} = 0.4545$$

$$\therefore \qquad N(\sigma_1 - k\sigma_3) = 2F(2106 + 31.25) = 4274.5F$$

From Eq. (4.18a),

$$4274.5F = \sigma_{yt} = 150 \times 10^6\ \text{Pa}$$

or $\qquad F = 34092\text{N}$

4.7 IDEALLY PLASTIC SOLID

If a rod of a ductile metal, such as mild steel, is tested under a simple uniaxial tension, the stress–strain diagram would be like the one shown in Fig. 4.9(a). As can be observed, the curve has several distinct regions. Part OA is linear, signifying that in this region, the strain is proportional to the stress. If a specimen is loaded within this limit and gradually unloaded, it returns to its original length

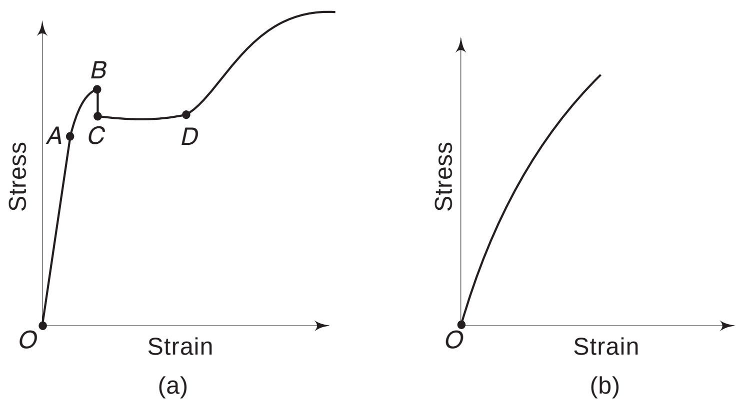

Fig. 4.9 *Stress–strain diagram for (a) Ductile material (b) Brittle material*

without any permanent deformation. This is the linear elastic region and point *A* denotes the limit of proportionality. Beyond *A*, the curve becomes slightly non-linear. However, the strain upto point *B* is still elastic. Point *B*, therefore, represents the elastic limit.

If the specimen is strained further, the stress drops suddenly (represented by point *C*) and thereafter the material yields at constant stress. After *D*, further straining is accompanied by increased stress, indicating work hardening. In the figure, the elastic region is shown exaggerated for clarity.

Most metals and alloys do not have a distinct yield point. The change from the purely elastic to the elastic-plastic state is gradual. Brittle materials, such as cast iron, titanium carbide or rock material, allow very little plastic deformation before reaching the breaking point. The stress-strain diagram for such a material would look like the one shown in Fig. 4.9(b).

In order to develop stress-strain relations during plastic deformation, the actual stress-strain diagrams are replaced by less complicated ones. These are shown in Fig. 4.10. In these, Fig. 4.10(a) represents a linearly elastic material, while Fig. 4.10(b) represents a material which is rigid (i.e. has no deformation) for stresses below σ_y and yields without limit when the stress level reaches the value σ_y. Such a material is called a rigid perfectly plastic material. Figure 4.10(c) shows the behaviour of a material which is rigid for stresses below σ_y and for stress levels above σ_y a linear work hardening characteristics is exhibited. A material exhibiting this characteristic behaviour is designated as rigid linear work hardening. Figure 4.10(d) and (e) represent respectively linearly elastic, perfectly plastic and linearly elastic–linear work hardening.

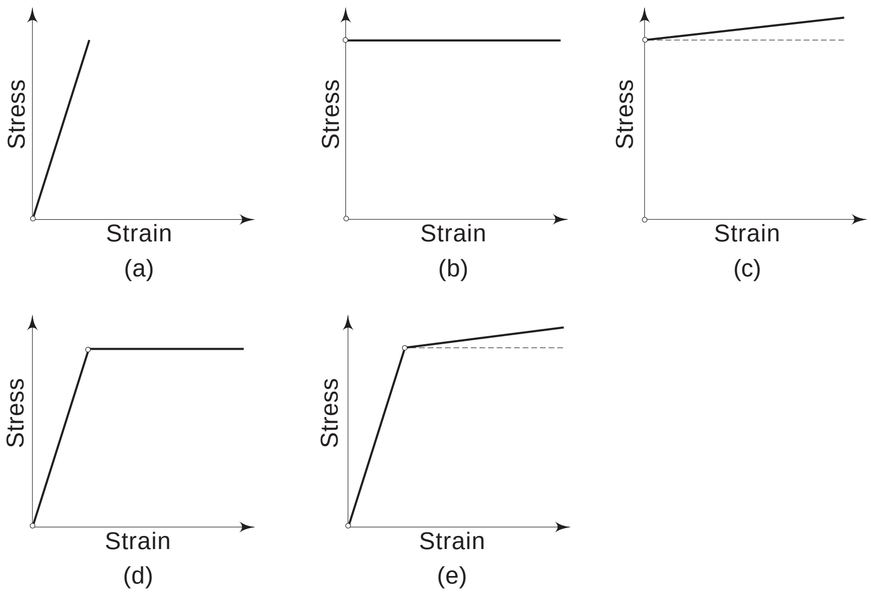

Fig. 4.10 *Ideal stress-strain diagram for a material that is (a) Linearly elastic (b) Rigid-perfectly plastic (c) Rigid-linear work hardening (d) Linearly elastic-perfectly plastic (e) Linearly elastic-linear work hardening*

In the following sections, we shall very briefly discuss certain elementary aspects of the stress-strain relations for an ideally plastic solid. It is assumed that the material behaviour in tension or compression is identical.

4.8 STRESS SPACE AND STRAIN SPACE

The state of stress at a point can be represented by the six rectangular stress components τ_{ij} $(i, j = 1, 2, 3)$. One can imagine a six-dimensional space called the stress space, in which the state of stress can be represented by a point. Similarly, the state of strain at a point can be represented by a point in a six- dimensional strain space. In particular, a state of plastic strain $\varepsilon_{ij}^{(p)}$ can be so represented. A history of loading can be represented by a path in the stress space and the corresponding deformation or strain history as a path in the strain space.

A basic assumption that is now made is that there exists a scalar function called a stress function or loading function, represented by $f(\tau_{ij}, \varepsilon_{ij}, K)$, which depends on the states of stress and strain, and the history of loading. The function $f = 0$ represents a closed surface in the stress space. The function f characterises the yielding of the material as follows:

As long as $f < 0$ no plastic deformation or yielding takes place; $f > 0$ has no meaning. Yielding occurs when $f = 0$. For materials with no work hardening characteristics, the parameter $K = 0$.

In the previous sections of this chapter, several yield criteria have been considered. These criteria were expressed in terms of the principal stresses $(\sigma_1, \sigma_2, \sigma_3)$ and the principal strains $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$. We have also observed that a material is said to be isotropic if the material properties do not depend on the particular coordinate axes chosen. Similarly, the plastic characteristics of the material are said to be isotropic if the yield function f depends only on the invariants of stress, strain and strain history. The isotropic stress theory of plasticity gives function f as an isotropic function of stresses alone. For such theories, the yield function can be expressed as $f(l_1, l_2, l_3)$ where l_1, l_2 and l_3 are the stress invariants. Equivalently, one may express the function as $f(\sigma_1, \sigma_2, \sigma_3)$. It is, therefore, possible to represent the yield surface in a three-dimensional space with coordinate axes σ_1, σ_2 and σ_3.

The Deviatoric Plane or the π Plane

In Section 4.2(a), it was stated that most metals can withstand considerable hydrostatic pressure without any permanent deformation. It has also been observed that a given state of stress can be uniquely resolved into a hydrostatic (or isotropic) state and a deviatoric (i.e. pure shear) state, i.e.

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} = \begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & P \end{bmatrix} = \begin{bmatrix} \sigma_1 - p & 0 & 0 \\ 0 & \sigma_2 - p & 0 \\ 0 & 0 & \sigma_3 - p \end{bmatrix}$$

or
$$[\sigma_i] = [p] + [\sigma_i^*], \qquad (i = 1, 2, 3) \tag{4.19}$$

where $$p = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$$

is the mean normal stress, and

$$\sigma_i^* = \sigma_i - p, \qquad (i = 1, 2, 3)$$

Consequently, the yield function will be independent of the hydrostatic state. For the deviatoric state, $l_1^* = 0$. According to the isotropic stress theory, therefore, the yield function will be a function of the second and third invariants of the devatoric state, i.e. $f(l_2^*, l_3^*)$. The equation

$$\sigma_1^* + \sigma_2^* + \sigma_3^* = 0 \tag{4.20}$$

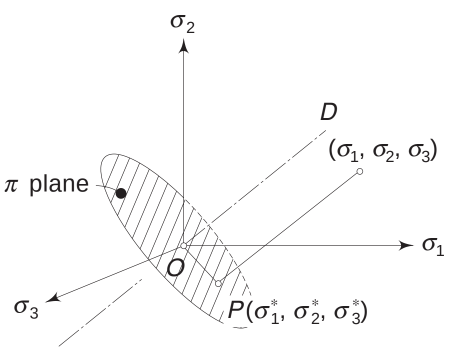

Fig. 4.11 *The π Plane*

represents a plane passing through the origin, whose normal *OD* is equally inclined (with direction cosines $1/\sqrt{3}$) to the axes σ_1, σ_2 and σ_3. This plane is called the deviatoric plane or the π plane. If the stress state $(\sigma_1^*, \sigma_2^*, \sigma_3^*)$ causes yielding, the point representing this state will lie in the π plane. This is shown by point *P* in Fig. 4.11. Since the addition or subtraction of an isotropic state does not affect the yielding process, point *P* can be moved parallel to *OD*. Hence, the yield function will represent a cylinder perpendicular to the π plane. The trace of this surface on the π plane is the yield locus.

4.9 GENERAL NATURE OF THE YIELD LOCUS

Since the yield surface is a cylinder perpendicular to the π plane, we can discuss its characteristics with reference to its trace on the π plane, i.e. with reference to the yield locus. Figure 4.12 shows the π plane and the projections of the σ_1, σ_2 and σ_3 axes on this plane as σ'_1, σ'_2 and σ'_3. These projections make an angle of 120° with each other.

Let us assume that the state (6, 0, 0) lies on the yield surface, i.e. the state $\sigma_1 = 6$, $\sigma_2 = 0$, $\sigma_3 = 0$, causes yielding. Since we have assumed isotropy, the states (0, 6, 0) and (0, 0, 6) also should cause yielding. Further, as we have assumed that the material behaviour in tension is identical to that in compression, the states (–6, 0, 0), (0, –6, 0) and (0, 0, –6) also cause yielding. Thus, appealing to isotropy and the property of the material in tension and compression, one point on the yield surface locates five other points. If we choose a general point (*a*, *b*, *c*) on the yield surface, this will generate 11 other (or a total of 12) points on the surface. These are (*a*, *b*, *c*) (*c*, *a*, *b*), (*b*, *c*, *a*), (*a*, *c*, *b*), (*c*, *b*, *a*) (*b*, *a*, *c*) and the remaining six are obtained by multiplying these by –1. Therefore, the yield locus is a symmetrical curve.

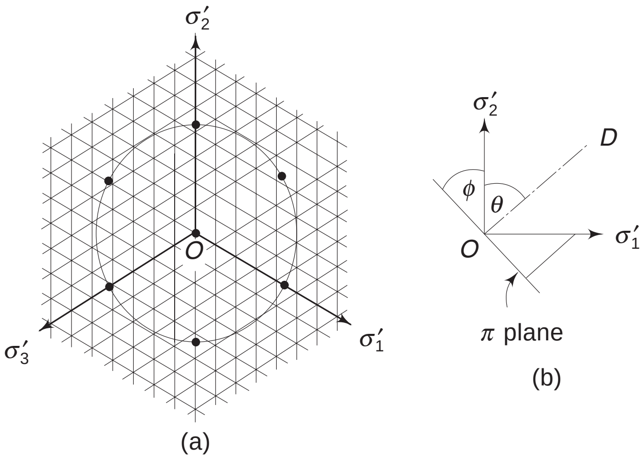

Fig. 4.12 *(a) The yield locus (b) Projection of π plane*

4.10 YIELD SURFACES OF TRESCA AND VON MISES

One of the yield conditions studied in Section 4.2 was stated by the maximum shear stress theory. According to this theory, if $\sigma_1 > \sigma_2 > \sigma_3$, the yielding starts when the maximum shear $\frac{1}{2}(\sigma_1 - \sigma_3)$ becomes equal to the maximum shear $\sigma_y/2$ in uniaxial tension yielding. In other words, yielding begins when $\sigma_1 - \sigma_3 = \sigma_y$. This condition is generally named after Tresca.

Let us assume that only σ_1 is acting. Then, yielding occurs when $\sigma_1 = \sigma_y$. The σ_1 axis is inclined at an angle of ϕ to its projection σ'_1 axis on the π plane, and $\sin\phi = \cos\theta = 1/\sqrt{3}$, [Fig. 4.12(b)]. Hence, the point $\sigma_1 = \sigma_y$ will have its projection on the π plane as $\sigma_y \cos\phi = \sqrt{2/3}\,\sigma_y$ along the σ'_1 axis. Similarly, other points on the π plane will be at distances of $\pm\sqrt{2/3}\,\sigma_y$ along the projections of σ_1, σ_2 and σ_3 axes on the π plane, i.e., along σ'_1, σ'_2, σ'_3 axes in Fig. 4.13. If σ_1, σ_2 and σ_3 are all acting (with $\sigma_1 > \sigma_2 > \sigma_3$), yielding occurs when $\sigma_1 - \sigma_3 = \sigma_y$. This defines a straight line joining points at a distance of σ_y along σ_1 and $-\sigma_3$ axes. The projection of this line on the π plane will be a straight line joining points at a distance of $\sqrt{2/3}\,\sigma_y$ along the σ'_1 and $-\sigma'_3$ axes, as shown by *AB* in Fig. 4.13. Consequently, the yield locus is a hexagon.

Another yield criterion discussed in Section 4.2 was the octahedral shearing stress or the distortion energy theory. According to this criterion, Eq. (4.4b), yielding occurs when

$$f(l_1, l_2, l_3) = f(l_1^2 - 3l_2) = \sigma_y^2 \tag{4.21}$$

Since a hydrostatic state of stress does not have any effect on yielding, one can deal with the deviatoric state (for which $l_1^* = 0$) and write the above condition as

$$f(l_2^*, l_3^*) = f(l_2^*) = -3\,l_2^* = \sigma_y^2 \tag{4.22}$$

The yield function can, therefore, be written as

$$f = l_2^* + \frac{1}{3}\sigma_y^2 = l_2^* + s^2 \tag{4.23}$$

where s is a constant. This yield criterion is known as the von Mises condition for yielding. The yield surface is defined by

$$I_2^* + s^2 = 0$$

or $$\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 - 3p^2 = -s^2 \tag{4.24}$$

The other alternative forms of the above expression are

$$(\sigma_1 - p)^2 + (\sigma_2 - p)^2 + (\sigma_3 - p)^2 = 2s^2 \tag{4.25}$$

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 6s^2 \tag{4.26}$$

Equation (4.25) can also be written as

$$\sigma_1^* + \sigma_2^* + \sigma_3^* = 2s^2 \tag{4.27}$$

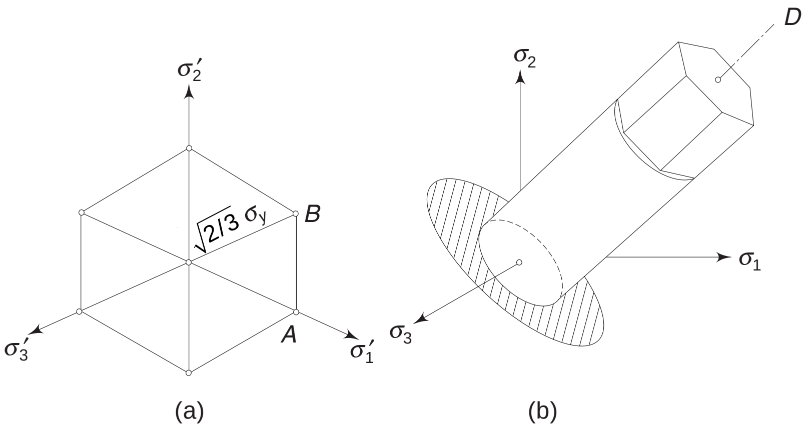

Fig. 4.13 *Yield surfaces of Tresca and von Mises*

This is the curve of intersection between the sphere $\sigma_1^2 + \sigma_2^2 + \sigma_3^2 = 2s^2$ and the π plane defined by $\sigma_1^* + \sigma_2^* + \sigma_3^* = 0$. This curve is, therefore, a circle with radius $\sqrt{2}s$ in the π plane. The yield surface according to the von Mises criterion is, therefore, a right circular cylinder. From Eq. (4.23)

$$s^2 = \frac{1}{3}\sigma_y^2, \qquad \text{or,} \qquad s = \frac{1}{\sqrt{3}}\sigma_y \tag{4.28}$$

Hence, the radius of the cylinder is $\sqrt{2/3}\,\sigma_y$ i.e. the cylinder of von Mises circumscribes Tresca's hexagonal cylinder. This is shown in Fig. 4.13.

4.11 STRESS–STRAIN RELATIONS (PLASTIC FLOW)

The yield locus that has been discussed so far defines the boundary of the elastic zone in the stress space. When a stress point reaches this boundary, plastic deformation takes place. In this context, one can speak of only the change in the plastic strain rather than the total plastic strain because the latter is the sum total of all plastic strains that have taken place during the previous strain history of the specimen. Consequently, the stress–strain relations for plastic flow relate the strain increments. Another way of explaining this is to realise that the process of plastic flow is irreversible; that most of the deformation work is transformed into heat and that the stresses in the final state depend on the strain path. Consequently, the equations governing plastic deformation cannot, in principle, be finite relations concerning stress and strain components as in the case of Hooke's law, but must be differential relations.

The following assumptions are made:

(i) The body is isotropic

(ii) The volumetric strain is an elastic strain and is proportional to the mean pressure ($\sigma_m = p = \sigma$)

$$\varepsilon = 3k\sigma$$

or

$$d\varepsilon = 3k d\sigma \tag{4.29}$$

(iii) The total strain increments $d\varepsilon_{ij}$ are made up of the elastic strain increments $d\varepsilon_{ij}^e$ and plastic strain increaments $d\varepsilon_{ij}^p$

$$d\varepsilon_{ij} = d\varepsilon_{ij}^e + d\varepsilon_{ij}^p \tag{4.30}$$

(iv) The elastic strain increments are related to stress components σ_{ij} through Hooke's law

$$d\varepsilon_{xx}^e = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$

$$d\varepsilon_{yy}^e = \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)]$$

$$d\varepsilon_{zz}^e = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)] \tag{4.31}$$

$$d\varepsilon_{xy}^e = d\gamma_{xy}^e = \frac{1}{G}\tau_{xy}$$

$$d\varepsilon_{yz}^e = d\gamma_{yz}^e = \frac{1}{G}\tau_{yz}$$

$$d\varepsilon_{zx}^e = d\gamma_{zx}^e = \frac{1}{G}\tau_{zx}$$

(v) The deviatoric components of the plastic strain increments are proportional to the components of the deviatoric state of stress

$$d\left[\varepsilon_{xx}^p - \frac{1}{3}(\varepsilon_{xx}^p + \varepsilon_{yy}^p + \varepsilon_{zz}^p)\right] = \left[\sigma_x - \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z)\right] d\lambda \tag{4.32}$$

where $d\lambda$ is the instantaneous constant of proportionality.

From (ii), the volumetric strain is purely elastic and hence

$$\varepsilon = \varepsilon_{xx}^e + \varepsilon_{yy}^e + \varepsilon_{zz}^e$$

But

$$\varepsilon = \varepsilon_{xx}^e + \varepsilon_{yy}^e + \varepsilon_{zz}^e + (\varepsilon_{xx}^p + \varepsilon_{yy}^p + \varepsilon_{zz}^p)$$

Hence,

$$\varepsilon_{xx}^p + \varepsilon_{yy}^p + \varepsilon_{zz}^p = 0 \tag{4.33}$$

Using this in Eq. (4.32)

$$d\varepsilon_{xx}^{p} = d\lambda \left[\sigma_x - \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z) \right]$$

Denoting the components of stress deviator by s_{ij}, the above equations and the remaining ones are

$$\begin{aligned} d\varepsilon_{xx}^{p} &= d\lambda\, s_{xx} \\ d\varepsilon_{yy}^{p} &= d\lambda\, s_{yy} \\ d\varepsilon_{zz}^{p} &= d\lambda\, s_{zz} \\ d\gamma_{xy}^{p} &= d\lambda\, s_{xy} \\ d\gamma_{yz}^{p} &= d\lambda\, s_{yz} \\ d\gamma_{zx}^{p} &= d\lambda\, s_{zx} \end{aligned} \tag{4.34}$$

Equivalently

$$d\varepsilon_{ij}^{p} = d\lambda\, s_{ij} \tag{4.35}$$

4.12 PRANDTL–REUSS EQUATIONS

Combining Eqs (4.30), (4.31) and (4.35)

$$d\varepsilon_{ij} = d\varepsilon_{ij}^{(e)} + d\lambda\, s_{ij} \tag{4.36}$$

where $d\varepsilon_{ij}^{(e)}$ is related to stress components through Hooke's law, as given in Eq. (4.31). Equations (4.30), (4.31) and (4.35) constitute the Prandtl–Reuss equations. It is also observed that the principal axes of stress and plastic strain increments coincide. It is easy to show that $d\lambda$ is non-negative. For this, consider the work done during the plastic strain increment

$$\begin{aligned} dW_p &= \sigma_x d\varepsilon_{xx}^{p} + \sigma_y d\varepsilon_{yy}^{p} + \sigma_z d\varepsilon_{zz}^{p} + \tau_{xy} d\gamma_{xy}^{p} + \tau_{yz} d\gamma_{yz}^{p} + \tau_{zx} d\gamma_{zx}^{p} \\ &= d\lambda \left(\sigma_x s_{xx} + \sigma_y s_{yy} + \sigma_z s_{zz} + \tau_{xy} s_{xy} + \tau_{yz} s_{yz} + \tau_{zx} s_{zx} \right) \\ &= d\lambda \left[\sigma_x (\sigma_x - p) + \sigma_y (\sigma_y - p) + \sigma_z (\sigma_z - p) + \tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2 \right] \end{aligned}$$

or $$dW_p = d\lambda \left[(\sigma_x - p)^2 + (\sigma_y - p)^2 + (\sigma_z - p)^2 + \tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2 \right]$$

i.e. $$dW_p = d\lambda T^2 \tag{4.37}$$

Since $dW_p \geq 0$

we have $d\lambda \geq 0$

If the von Mises condition is applied, from Eqs (4.23) and (4.35)

$$dW_p = d\lambda 2s^2$$

or
$$d\lambda = \frac{dW_p}{2s^2} \tag{4.38}$$

i.e $d\lambda$ is proportional to the increment of plastic work.

4.13 SAINT VENANT–VON MISES EQUATIONS

In a fully developed plastic deformation, the elastic components of strain are very small compared to plastic components. In such a case

$$d\varepsilon_{ij} \approx d\varepsilon_{ij}^{p}$$

and this gives the equations of the Saint Venant–von Mises theory of plasticity in the form

$$d\varepsilon_{ij} = d\lambda\, s_{ij} \tag{4.39}$$

Expanding this

$$d\varepsilon_{xx} = \frac{2}{3} d\lambda \left[\sigma_x - \frac{1}{2}\left(\sigma_y + \sigma_z\right)\right]$$

$$d\varepsilon_{yy} = \frac{2}{3} d\lambda \left[\sigma_y - \frac{1}{2}\left(\sigma_z + \sigma_x\right)\right]$$

$$d\varepsilon_{zz} = \frac{2}{3} d\lambda \left[\sigma_z - \frac{1}{2}\left(\sigma_x + \sigma_y\right)\right] \tag{4.40}$$

$$d\gamma_{xy} = d\lambda \tau_{xy}$$
$$d\gamma_{yz} = d\lambda \tau_{yz}$$
$$d\gamma_{zx} = d\lambda \tau_{zx}$$

The above equations are also called Levy–Mises equations. In this case, it should be observed that the principal axes of strain increments coincide with the axes of the principal stresses.

Problems

4.1 Figure 4.14 shows three elements *a*, *b*, *c* subjected to different states of stress. Which one of these three, do you think, will yield first according to
(i) the maximum stress theory?
(ii) the maximum strain theory?

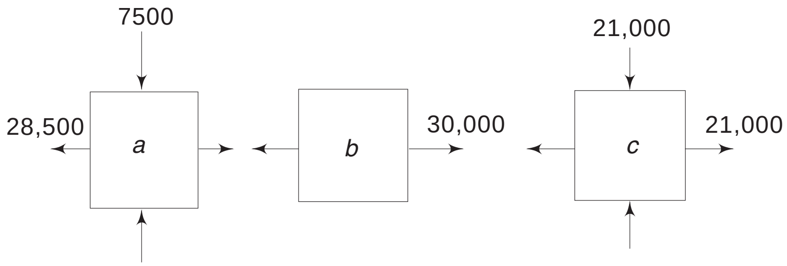

Fig. 4.14 *Problem 4.1*

(iii) the maximum shear stress theory?

Poisson's ratio $\nu = 0.25$ [*Ans.* (i) *b*, (ii) *a*, (iii) *c*]

4.2 Determine the diameter of a cold-rolled steel shaft, 0.6 m long, used to transmit 50 hp at 600 rpm. The shaft is simply supported at its ends in bearings. The shaft experiences bending owing to its own weight also. Use a factor of safety 2. The tensile yield limit is 280×10^3 kPa (2.86×10^3 kgf/cm^2) and the shear yield limit is 140×10^3 kPa (1.43×10^3 kgf/cm^2). Use the maximum shear stress theory. [*Ans.* $d = 3.6$ cm]

4.3 Determine the diameter of a ductile steel bar (Fig 4.15) if the tensile load F is 35,000 N and the torsional moment T is 1800 Nm. Use a factor of safety $N = 1.5$.

$E = 207 \times 10^6$ kPa $\left(2.1 \times 10^6 \text{ kgf/cm}^2\right)$ and σ_{yp} is 207,000 kPa (2100 kgf/cm^2).

Use the maximum shear stress theory. [*Ans.* $d = 4.1$ cm]

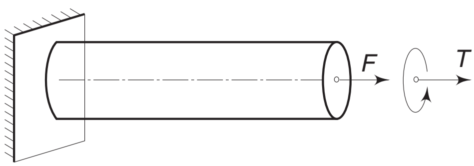

Fig. 4.15 *Problem 4.3*

4.4 For the problem discussed in Problem 4.3, determine the diameter according to Mohr's theory if $\sigma_{yt} = 207$ MPa, $\sigma_{yc} = 310$ MPa. The factor of safety $N = 1.5$; $F = 35{,}000$ N and $T = 1800$ Nm. [*Ans.* $d = 4.2$ cm]

4.5 At a point in a steel member, the state of stress is as shown in Fig. 4.16. The tensile elastic limit is 413.7 kPa. If the shearing stress at the point is 206.85 kPa, when yielding starts, what is the tensile stress σ at the point (a) according to the maximum shearing stress theory, and (b) according to the octahedral shearing stress theory?

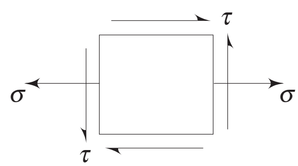

Fig. 4.16 *Problem 4.5*

[*Ans.* (a) zero; (b) 206.85 kPa (2.1 kgf/cm^2)]

4.6 A torque T is transmitted by means of a system of gears to the shaft shown in Fig. 4.17. If T = 2500Nm (25,510 kgf cm), R = 0.08 m, a = 0.8 m and b = 0.1 m, determine the diameter of the shaft, using the maximum shear stress theory. $\sigma_y = 290000$ kPa. The factor of safety is 2. Note that when a torque is being transmitted, in addition to the tangential force, there occurs a radial force equal to 0.4F, where F is the tangential force. This is shown in Fig. 4.17(b).

Hint: The forces F and 0.4F acting on the gear A are shown in Fig. 4.17(b). The reactions at the bearings are also shown. There are two bending moments—one in the vertical plane and the other in the horizontal plane. In the vertical plane, the maximum moment is $\dfrac{(0.4Fab)}{(a+b)}$;

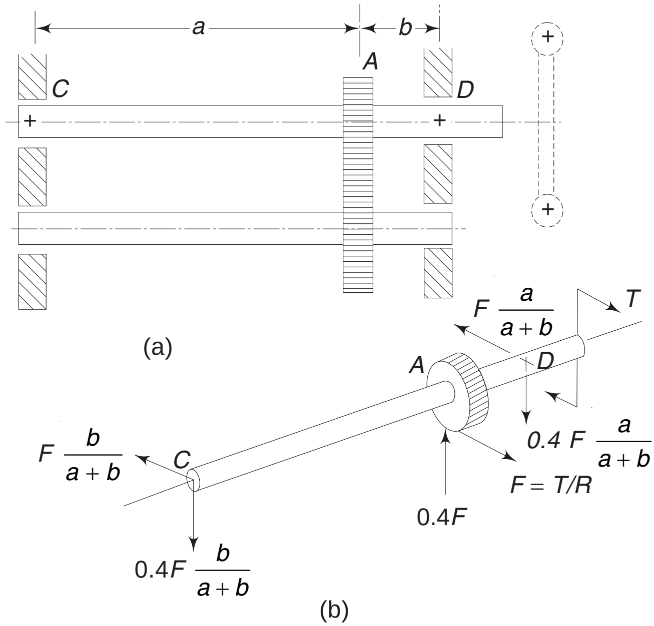

Fig. 4.17 *Problem 4.6*

in the horizontal plane the maximum moment is $\frac{(Fab)}{(a+b)}$; both these maximums occur at the gear section *A*. The resultant bending is

$$(M)_{max} = \left[\left(\frac{0.4\,Fab}{a+b}\right)^2 + \left(\frac{Fab}{a+b}\right)^2\right]^{1/2}$$

$$= 1.08F\,\frac{ab}{a+b}$$

The critical point to be considered is the circumferential point on the shaft subjected to this maximum moment. [*Ans*. $d \approx 65$ mm]

4.7 If the material of the bar in Problem 4.4 has $\sigma_{yt} = 207 \times 10^6$ Pa and $\sigma_{yc} = 517 \times 10^6$ Pa determine the diameter of the bar according to Mohr's theory of failure. The other conditions are as given in Problem 4.4. [*Ans*. $d = 4.6$ cm]

CHAPTER 5

Energy Methods

5.1 INTRODUCTION

In Chapters 1 and 2, attention was focussed on the analysis of stress and strain at a point. Except for the condition that the material we considered was a continuum, the shape or size of the body as a whole was not considered. In Chapter 3, the stresses and strains at a point were related through the material or the constitutive equations. Here too, the material properties rather than the behaviour of the body as such was not considered. Chapter 4, on the theory of failure, also discussed the critical conditions to impend failure at a point. In this chapter, we shall consider the entire body or structural member or machine element, along with the forces acting on it. Hooke's law will relate the force acting on the body to the displacement. When the body deforms under the action of the externally applied forces, the work done by these forces is stored as strain energy inside the body, which can be recovered when the latter is elastic in nature. It is assumed that the forces are applied gradually.

The strain energy methods are extremely important for the solution of many problems in the mechanics of solids and in structural analysis. Many of the theorems developed in this chapter can be used with great advantage to solve displacement problems and statically indeterminate structures and frameworks.

5.2 HOOKE'S LAW AND THE PRINCIPLE OF SUPERPOSITION

We have observed in Chapter 3 that the rectangular stress components at a point can be related to the rectangular strain components at the same point through a set of linear equations that were designated as the generalised Hooke's Law. In this chapter, however, we shall state Hooke's law as applicable to the elastic body as a whole, i.e. relate the complete system of forces acting on the body to the deformation of the body as a whole. The law asserts that 'deflections are proportional to the forces which produce them'. This is a very general assertion without any restriction as to the shape or size of the loaded body.

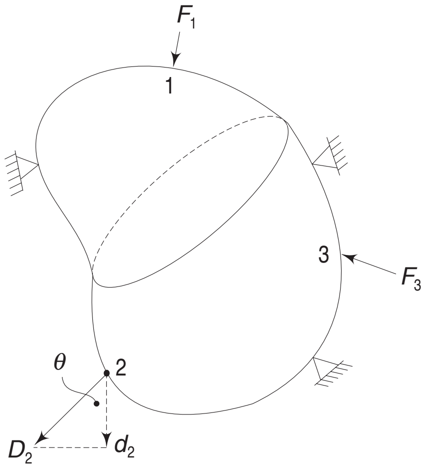

Fig. 5.1 *Elastic solid and Hooke's law*

In Fig. 5.1, a force F_1 is applied at point 1, and in consequence, point 2 undergoes a displacement or a deflection, which according to Hooke's law, is proportionate to F_1. This deflection of point 2 may take place in a direction which is quite different from that of F_1. If D_2 is the actual deflection, we have

$$D_2 = k_{21}F_1$$

where k_{21} is some proportionality constant.

When F_1 is increased, D_2 also increases proportionately. Let d_2 be the component of D_2 in a specified direction. If θ is the angle between D_2 and d_2

$$d_2 = D_2 \cos\theta = k_{21} \cos\theta \, F_1$$

If we keep θ constant, i.e. if we fix our attention on the deflection in a specified direction, then

$$d_2 = a_{21}F_1$$

where a_{21} is a constant. Therefore, one can consider the displacement of point 2 in any specified direction and apply Hooke's Law. Let us consider the vertical component of the deflection of point 2. If d_2 is the vertical component, then Hooke's law asserts that

$$d_2 = a_{21}F_1 \tag{5.1}$$

where a_{21} is a constant called the 'influence coefficient' for vertical deflection at point 2 due to a force applied in the specified direction (that of F_1) at point 1. If F_1 is a unit force, then a_{21} is the actual value of the vertical deflection at 2. If a force equal and opposite to F_1 is applied at 1, then a deflection equal and opposite to the earlier deflection takes place. If several forces, all having the direction of F_1, are applied simultaneously at 1, the resultant vertical deflection which they produce at 2 will be the resultant of the deflections which they would have produced if applied separately. This is the principle of superposition.

Consider a force F_3 acting alone at point 3, and let d_2' be the vertical component of the deflection of 2. Then, according to Hooke's Law, as stated by Eq. (5.1)

$$d_2' = a_{23}F_3 \tag{5.2}$$

where a_{23} is the influence coefficient for vertical deflection at point 2 due to a force applied in the specified direction (that of F_3) at point 3. The question that we now examine is whether the principle of superposition holds true to two or more forces, such as F_1 and F_3, which act in different directions and at different points.

Let F_1 be applied first, and then F_3. The vertical deflection at 2 is

$$d_2 = a_{21}F_1 + a_{23}'F_3 \tag{5.3}$$

where a'_{23} may be different from a_{23}. This difference, if it exists, is due to the presence of F_1 when F_3 is applied. Now apply $-F_1$. Then

$$= a_{21}F_1 + a'_{23}F_3 - a'_{21}F_1$$

a'_{21} may be different from a_{21}, since F_3 is acting when $-F_1$ is applied. Only F_3 is acting now. If we apply $-F_3$, the deflection finally becomes

$$d''_2 = a_{21}F_1 + a'_{23}F_3 - a'_{21}F_1 - a_{23}F_3 \tag{5.4}$$

Since the elastic body is not subjected to any force now, the final deflection given by Eq. (5.4) must be zero. Hence,

$$a_{21}F_1 + a'_{23}F_3 - a'_{21}F_1 - a_{23}F_3 = 0$$

i.e. $$(a_{21} - a'_{21})F_1 = (a_{23} - a'_{23})F_3$$

or $$\frac{a_{21} - a'_{21}}{F_3} = \frac{a_{23} - a'_{23}}{F_1} \tag{5.5}$$

The difference $a_{21} - a'_{21}$, if it exists, must be due to the action of F_3. Hence, the left-hand side is a function of F_3 alone. Similarly, if the difference $a_{23} - a'_{23}$ exists, it must be due to the action of F_1 and, therefore, the right-hand side must be a function F_1 alone. Consequently, Eq. (5.5) becomes

$$\frac{a_{21} - a'_{21}}{F_3} = \frac{a_{23} - a'_{23}}{F_1} = k \qquad d''_2 \tag{5.6}$$

where k is a constant independent of F_1 and F_3. Hence

$$a'_{23} = a_{23} - kF_1$$

Substituting this in Eq. (5.3)

$$d_2 = a_{21}F_1 + a_{23}F_3 - kF_1F_3$$

The last term on the right-hand side in the above equation is non-linear, which is contradictory to Hooke's law, unless k vanishes. Hence, $k = 0$, and

$$a_{23} = a'_{23} \quad \text{and} \quad a_{21} = a'_{21}$$

The principle of superposition is, therefore, valid for two different forces acting at two different points. This can be extended by induction to include a third or any number of other forces. This means that the deflection at 2 due to any number of forces, including force F_2 at 2 is

$$d_2 = a_{21}F_1 + a_{22}F_2 + a_{23}F_3 + \dots \tag{5.7}$$

5.3 CORRESPONDING FORCE AND DISPLACEMENT OR WORK-ABSORBING COMPONENT OF DISPLACEMENT

Consider an elastic body which is in equilibrium under the action of forces F_1, F_2, F_3, . . . The forces of reaction at the points of support will also be considered as applied forces. This is shown in Fig. 5.2.

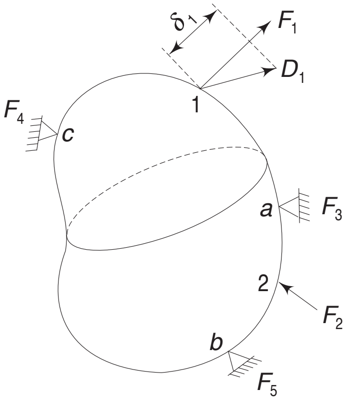

Fig. 5.2 *Corresponding forces and displacements*

The displacement d_1 in a specified direction at point 1 is given by Eq. (5.7). If the actual displacement is D_1 and takes place in a direction as shown in Fig. (5.2), then the component of this displacement in the direction of force F_1 is called the corresponding displacement at point 1. This corresponding displacement is denoted by δ_1. At every loaded point, a corresponding displacement can be identified. If the points of support a, b and c do not yield, then at these points the corresponding displacements are zero. One can apply Hooke's law to these corresponding displacements and obtain from Eq. (5.7)

$$\delta_1 = a_{11}F_1 + a_{12}F_2 + a_{13}F_3 + \dots$$
$$\delta_2 = a_{21}F_1 + a_{22}F_2 + a_{23}F_3 + \dots \text{ etc.} \quad (5.8)$$

where a_{11}, a_{12}, a_{13}, . . ., are the influence coefficients of the kind discussed earlier. The corresponding displacement is also called the work-absorbing component of the displacement.

5.4 WORK DONE BY FORCES AND ELASTIC STRAIN ENERGY STORED

Equations (5.8) show that the displacements δ_1, δ_2, . . .etc., depend on all the forces F_1, F_2, . . ., etc. If we slowly increase the forces F_1, F_2, . . ., etc. from zero to their full magnitudes, the deflections also increase similarly. For example, when the forces F_1, F_2, . . ., etc. are one-half of their full magnitudes, the deflections are

$$\frac{1}{2}\delta_1 = a_{11}\left(\frac{1}{2}F_1\right) + a_{12}\left(\frac{1}{2}F_2\right) + \dots,$$
$$\frac{1}{2}\delta_2 = a_{21}\left(\frac{1}{2}F_1\right) + a_{22}\left(\frac{1}{2}F_2\right) + \dots, \text{ etc.,}$$

i.e. the deflections reached are also equal to half their full magnitudes. Similarly, when F_1, F_2, . . ., etc. reach two-thirds of their full magnitudes, the deflections reached are also equal to two-thirds of their full magnitudes. Assuming that the forces are increased in constant proportion and the increase is gradual, the work done by F_1 at its point of application will be

$$W_1 = \frac{1}{2}F_1\delta_1$$
$$= \frac{1}{2}F_1(a_{11}F_1 + a_{12}F_2 + a_{13}F_3 + \dots) \quad (5.9)$$

Similar expressions hold good for other forces also. The total work done by external forces is, therefore, given by

$$W_1 + W_2 + W_3 + \dots = \frac{1}{2}(F_1\delta_1 + F_2\delta_2 + F_3\delta_3 + \dots)$$

If the supports are rigid, then no work is done by the support reactions. When the forces are gradually reduced to zero, keeping their ratios constant, negative work will be done and the total work will be recovered. This shows that the work done is stored as potential energy and its magnitude should be independent of the order in which the forces are applied. If it were not so, it would be possible to store or extract energy by merely changing the order of loading and unloading. This would be contradictory to the principle of conservation of energy.

The potential energy that is stored as a consequence of the deformation of any elastic body is termed elastic strain energy. If F_1, F_2, F_3 are the forces in a particular configuration and δ_1, δ_2, δ_3 are the corresponding displacements then the elastic strain energy stored is

$$U = \frac{1}{2}(F_1\delta_1 + F_2\delta_2 + F_3\delta_3 + \ldots) \tag{5.10}$$

It must be noted that though this expression has been obtained on the assumption that the forces F_1, F_2, F_3 . . ., are increased in constant proportion, the conservation of energy principle and the superposition principle dictate that this expression for U must hold without restriction on the manner or order of the application of these forces.

5.5 RECIPROCAL RELATION

It is easy to show that the influence coefficient a_{12} in Eq. (5.8) is equal to the influence coefficient a_{21}. In general, $a_{ij} = a_{ji}$. To show this, consider a force F_1 applied at point 1, and let δ_1 be the corresponding displacement. The energy stored is

$$U_1 = \frac{1}{2}F_1\delta_1 = \frac{1}{2}a_{11}F_1^2$$

since $\delta_1 = a_{11}F_1$

Next, apply force F_2 at point 2. The corresponding deflection at point 2 is $a_{22}F_2$ and that at point 1 is $a_{12}F_2$. During this displacement, force F_1 is fully acting and hence, the additional energy stored is

$$U_2 = \frac{1}{2}F_2(a_{22}F_2) + F_1(a_{12}F_2)$$

The total elastic energy stored is therefore

$$U = U_1 + U_2 = \frac{1}{2}a_{11}F_1^2 + \frac{1}{2}a_{22}F_2^2 + a_{12}F_1F_2$$

Now, if F_2 is applied before F_1, the elastic energy stored is

$$U' = \frac{1}{2}a_{22}F_2^2 + \frac{1}{2}a_{11}F_1^2 + a_{21}F_1F_2$$

Since the elastic energy stored is independent of the order of application of F_1 and F_2, U and U' must be equal. Consequently,

$$a_{12} = a_{21} \tag{5.11a}$$

or in general

$$a_{ij} = a_{ji} \tag{5.11b}$$

The result expressed in Eq. (5.11b) has great importance in the mechanics of solids, as shown in the next section.

One can obtain an expression for the elastic strain energy in terms of the applied forces, using the above reciprocal relationship. From Eq. (5.10)

$$U = \frac{1}{2}(F_1\delta_1 + F_2\delta_2 + \ldots + F_n\delta_n)$$

$$= \frac{1}{2}F_1(a_{11}F_1 + a_{12}F_2 + \ldots + a_{1n}F_n)$$

$$+ \ldots \frac{1}{2} F_n(a_{n1}F_1 + a_{n2}F_2 + \ldots + a_{nn}F_n)$$

$$U = \frac{1}{2}(a_{11}F_1^2 + a_{22}F_2^2 + \ldots + a_{nn}F_n^2)$$

$$+ \frac{1}{2}(a_{12}F_1F_2 + a_{13}F_1F_3 + \ldots + a_{1n}F_1F_n + \ldots)$$

$$= \frac{1}{2}\Sigma(a_{11}F_1^2) + \Sigma(a_{12}F_1F_2) \tag{5.12}$$

5.6 MAXWELL–BETTI–RAYLEIGH RECIPROCAL THEOREM

Consider two systems of forces $F_1, F_2, \ldots$, and $F_1', F_2', \ldots$, both systems having the same points of application and the same directions. Let $\delta_1, \delta_2, \ldots$, be the corresponding displacements caused by $F_1, F_2, \ldots$, and $\delta_1', \delta_2', \ldots$, the corresponding displacements caused by $F_1', F_2', \ldots$, Then, making use of the reciprocal relation given by Eq. (5.11) we have

$$F_1'\delta_1 + F_2'\delta_2 + \ldots + F_n'\delta_n$$

$$= F_1'(a_{11}F_1 + a_{12}F_2 + \ldots + a_{1n}F_n)$$

$$+ F_2'(a_{21}F_1 + a_{22}F_2 + \ldots + a_{2n}F_n)$$

$$+ \ldots + F_n'(a_{n1}F_1 + a_{n2}F_2 + \ldots + a_{nn}F_n)$$

$$= a_{11}F_1F_1' + a_{22}F_2F_2' + a_{nn}F_nF_n'$$

$$+ a_{12}(F_1'F_2 + F_2'F_1) + a_{13}(F_1'F_3 + F_3'F_1)$$

$$+ \ldots + a_{1n}(F_1'F_n + F_n'F_1) \tag{5.13}$$

The symmetry of the expressions between the primed and unprimed quantities in the above expression shows that it is equal to

$$F_1\delta_1' + F_2\delta_2' + \ldots + F_n\delta_n'$$

i.e. $$F_1\delta_1' + F_2\delta_2' + \ldots = F_1'\delta_1 + F_2'\delta_2 + \ldots \tag{5.14}$$

In words:

'The forces of the first system ($F_1, F_2, \ldots$, etc.) acting through the corresponding displacements produced by any second system ($F_1', F_2', \ldots$, etc.) do the same amount of work as that done by the second system of forces acting through the corresponding displacements produced by the first system of forces'.

This is the reciprocal theorem of Maxwell, Betti and Rayleigh.

5.7 GENERALISED FORCES AND DISPLACEMENTS

In the above discussions, F_1 F_2, . . . , etc. represented concentrated forces and δ_1, δ_2, . . . , etc. the corresponding linear displacements. It is possible to extend the term 'force' to include not only a concentrated force but also a bending moment or a torque. Similarly, the term 'displacement' may mean linear or angular displacement. Consider, for example, the elastic body shown in Fig. 5.3, subjected to a concentrated force F_1 at point 1 and a couple $F_2 = M$ at point 2. δ_1 will now stand for the corresponding linear displacement of point 1 and δ_2 for the corresponding angular rotation of point 2. If F_1 is a unit force acting alone, then a_{11}, the influence coefficient, gives the linear displacement of point 1 corresponding to the direction of F_1. Similarly, a_{12} stands for the corresponding linear displacement of point 1 caused by a unit couple F_2 applied at point 2. a_{21} gives the corresponding angular rotation of point 2 caused by a unit concentrated force F_1 at point 1.

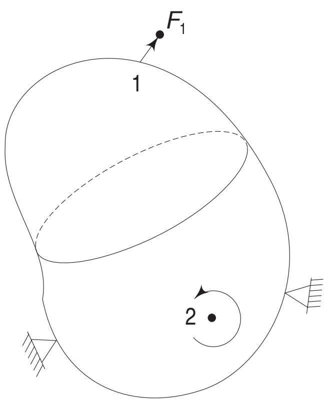

Fig. 5.3 *Generalised forces and displacements*

The reciprocal relation $a_{12} = a_{21}$ can also be interpreted appropriately. For example, making reference to Fig. 5.3, the above relation reveals that the linear displacement at point 1 in the direction of F_1 caused by a unit couple acting alone at point 2, is equal to the angular rotation at point 2 in the direction of the moment F_2 caused by a unit load acting alone at point 1. This fact will be demonstrated in the next few examples.

With the above generalised definitions for forces and displacements, the work done when the forces are gradually increased from zero to their full magnitudes is given by

$$W = \frac{1}{2}\left(F_1\delta_1 + F_2\delta_2 + \ldots + F_n\delta_n\right)$$

The reciprocal theorem of Maxwell, Betti and Rayleigh can also be given wider meaning with these extended definitions.

Example 5.1 *Consider a cantilever loaded by unit concentrated forces, as shown in Figs. 5.4(a) and (b). Check the deflections at points 1 and 2.*

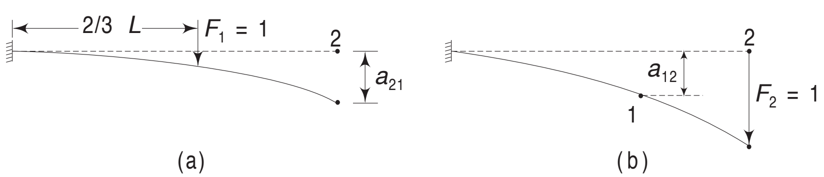

Fig. 5.4 *Example 5.1*

Solution In Fig. 5.4(a), the unit load F_1 acts at point 1. As a result, the deflection of point 2 is a_{21}. In Fig. 5.4(b) the unit load F_2 acts at point 2 and as a result, the deflection of point 1 is a_{12}. The reciprocal relation conveys that these two deflections are equal. If L is the length of the cantilever and if point 1 is at a distance of $\frac{2}{3}L$ from the fixed end, we have from elementary strength of materials

δ_2 due to F_1 = deflection at 1 due to F_1 + deflection due to slope

$$= \frac{8L^3}{81\,EI} + \frac{4L^3}{54\,EI}$$

δ_1 due to F_2 = deflection at 1 due to a unit load at 1 + deflection at 1 due to a moment (L/3) at 1

$$= \frac{8L^3}{81\,EI} + \frac{4L^3}{54\,EI}$$

Example 5.2 *Consider a cantilever beam subjected to a concentrated force F at point 1 (Fig 5.5). Let us determine the curve of deflection for the beam.*

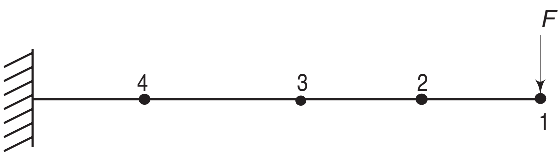

Fig. 5.5 *Example 5.2*

Solution One obvious method would be to use a travelling microscope and take readings at points 2, 3, 4, etc. These readings would be very small and consequently, errors would creep in. On the other hand, the reciprocal relation can be used to obtain this curve of deflection more accurately. The deflection at 2 due to F at 1 is the same as the deflection at 1 due to F at 2, i.e. $a_{21} = a_{12}$. Similarly, the deflection at 3 due to F at 1 is the same as the deflection at 1 due to F at 3, i.e. $a_{31} = a_{13}$. Hence, one observes the deflections at 1 as F is moved along the beam to get the required information.

Example 5.3 *The cantilever beam shown in Fig. 5.6(a) is subjected to a bending moment $M = F_1$ at point 1, and in Fig. 5.6(b), it is subjected to a concentrated load $P = F_2$ at point 2. Point 2 is 2/3 L from the fixed end. Verify the reciprocal theorem.*

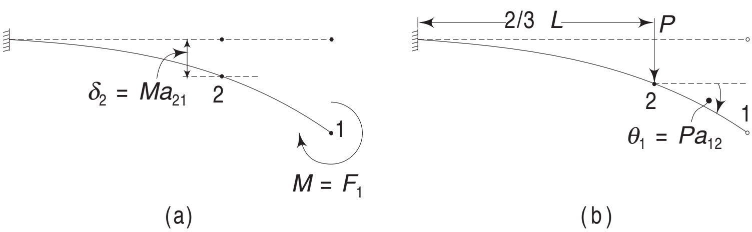

Fig. 5.6 *Example 5.3*

Solution From elementary strength of materials the deflection at point 2 due to the moment M at point 1 is

$$\delta_2 = M\left(\frac{2}{3}L\right)^2 \frac{1}{2EI} = \frac{2ML^2}{9EI}$$

The slope (angular displacement) at point 1 due to the concentrated force P at point 2 is

$$\theta_1 = P\left(\frac{2}{3}L\right)^2 \frac{1}{2EI} = \frac{2PL^2}{9EI}$$

Hence, the work done by M through the displacement (angular displacement) produced by P is equal to

$$M\theta_1 = \frac{2MPL^2}{9EI}$$

This is equal to the work done by P acting through the displacement produced by the moment M.

Example 5.4 *Determine the change in volume of an elastic body subjected to two equal and opposite forces, as shown. The distance between the points of application is h and the elastic constants for the material are E and ν, (Fig. 5.7).*

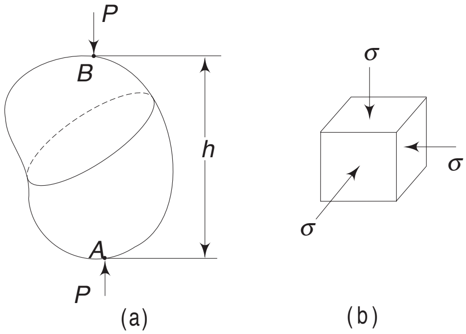

Fig. 5.7 *Example 5.4*

Solution This is a very general problem, the solution of which is apparently difficult. However, we can get a solution very easily by applying the reciprocal theorem. Let the elastic body be subjected to a hydrostatic pressure of value σ. Every volume element will be in a state of hydrostatic (isotropic) stress. Consequently, the unit contraction in any direction from Fig. 5.7(b) is

$$\varepsilon = \frac{\sigma}{E} - 2\nu\frac{\sigma}{E} = (1-2\nu)\frac{\sigma}{E}$$

The two points of application A and B, therefore, move towards each other by a distance.

$$\Delta h = h\,(1-2\nu)\frac{\sigma}{E}$$

Now we have two systems of forces:

System 1	Force	P
	Volume change	ΔV

System 2	Force	σ
	Distance change	Δh

From the reciprocal theorem

$$P\,\Delta h = \sigma\,\Delta V$$

or
$$\Delta V = \frac{P}{\sigma}\Delta h$$

$$= \frac{Ph}{E}(1-2\nu)$$

If ν is equal to 0.5, the change in volume is zero.

5.8 BEGG'S DEFORMETER

In this section, we shall demonstrate the application of the reciprocal theorem to a problem in experimental mechanics. Figure 5.8 shows a structural member subjected to a force P at point E. It is required to determine the forces of reaction at point B. The reaction forces are V, H and M and these make the displacements (vertical, horizontal and angular) at B equal to zero. A theoretical analysis is quite difficult for an odd structure like the one shown. The reactions at the other supports also are such that the displacement at these supports are zero. To determine V at B we proceed as follows.

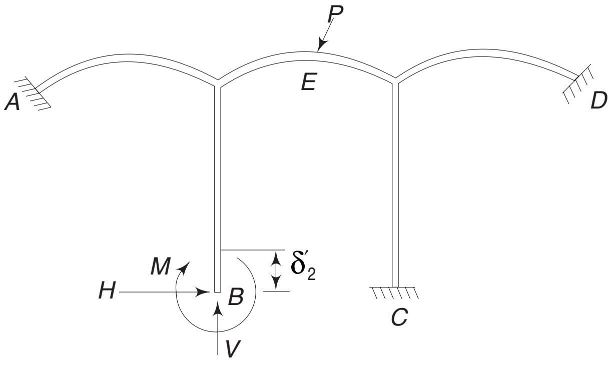

Fig. 5.8 *Reactions due to force P*

A known vertical displacement δ_2' is imposed at B, keeping A, C, D fixed and preventing angular rotation and horizontal displacement at B. The corresponding displacement at E (i.e. displacement in the direction of P) is measured. Let this be δ_1'. During the vertical displacement of B, the forces V', M' and H' that are induced at B are not measured. The two systems involved in the reciprocal theorem are as follows:

System 1 Specified

Forces V, H, M at B (unknown) and other reactive forces at A, C, D (also unknown), P at E (known)

Corresponding displacements 0, 0, 0 at B; 0, 0, 0 at A, C and D; δ_1 (unknown) at E.

System 2 Experimental

Forces V', H', M' at B (unknown) and other reactive forces at A, C, D (all unknown); 0 at E (i.e. point E not loaded)

Corresponding displacements δ_2', 0, 0 at B; 0, 0, 0 at A, C and D; δ_1' at E
Applying the reciprocal theorem

$$(V \cdot \delta_2') + (H \cdot 0) + (M \cdot 0) + 0 + (P \cdot \delta_1')$$
$$= (V' \cdot 0) + (H' \cdot 0) + (M' \cdot 0) + 0 + (0 \cdot \delta_1)$$

i.e.-
$$V = -P\frac{\delta_1'}{\delta_2'} \tag{5.15}$$

Since δ_2' is the known displacement imposed at B and δ_1' is the corresponding displacement at E that is experimentally measured, the value of V can be determined. It is necessary to remember that the corresponding displacement δ_1' at E is positive when it is in the direction of P.

To determine H at B, we proceed as above. A known horizontal displacement δ_2' is imposed at B, with all other displacements being kept zero. The corresponding displacement δ_1' at E is measured. The result is

$$H = -P\frac{\delta_1'}{\delta_2'}$$

To determine M at B, a known amount of small rotation θ' is imposed at B, keeping all other displacements zero. The corresponding displacement δ_1' resulting at E is measured. The reciprocal theorem again gives

$$M = -P\frac{\delta_1'}{\theta'}$$

5.9 FIRST THEOREM OF CASTIGLIANO

From Eq. (5.12), the expression for the elastic strain energy is

$$U = \frac{1}{2}\left(a_{11}F_1^2 + a_{22}F_2^2 + \ldots + a_{nn}F_n^2\right)$$
$$+ \left(a_{12}F_1F_2 + a_{13}F_1F_3 + \ldots + a_{1n}F_1F_n\right) + \ldots$$

In the above expression, F_1, F_2, etc. are the generalised forces, i.e. concentrated loads, moments or torques. a_{11}, a_{12}, . . . , etc. are the corresponding influence coefficients. The rate at which U increases with F_1 is given by $\frac{\partial U}{\partial F_1}$. From the above expression for U,

$$\frac{\partial U}{\partial F_1} = a_{11}F_1 + a_{12}F_2 + a_{13}F_3 + \ldots + a_{1n}F_n$$

This is nothing but the corresponding displacement at F_1, Eq. (5.8). Hence, if δ_1 stands for the generalised displacement (linear or angular) corresponding to the generalised force F_1, then

$$\frac{\partial U}{\partial F_1} = \delta_1 \tag{5.16}$$

In exactly the same way, one can show that

$$\frac{\partial U}{\partial F_2} = \delta_2, \quad \frac{\partial U}{\partial F_3} = \delta_3, \ldots, \text{etc.}$$

That is to say, 'the partial differential coefficient of the strain energy function with respect to F_r gives the displacement corresponding with F_r'. This is Castigliano's first theorem. In the form derived in Eq. (5.16), the theorem is applicable to only linearly elastic bodies, i.e. bodies satisfying Hooke's Law (see Sec. 5.15).

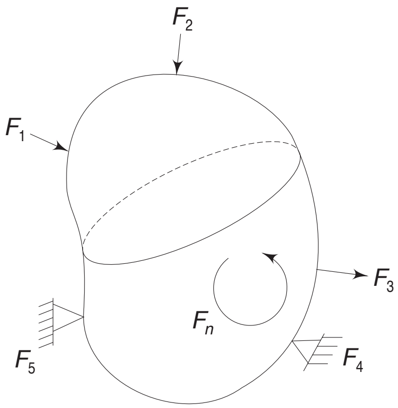

Fig. 5.9 *Elastic body in equilibrium under forces F_1, F_2, etc.*

This theorem is extremely useful in determining the displacements of structures as well as in the solutions of many statically indeterminate structures. Several examples will illustrate these subsequently. We can give an alternative proof for this theorem as follows:

Consider an elastic system in equilibrium under the force F_1, F_2, . . . F_n, etc. (Fig. 5.9). Some of these are concentrated loads and some are couples and torques. Let the strain energy stored be U. Now increase one of the forces, say F_n, by ΔF_n and as a result the strain energy increases to $U + \Delta U$, where

$$\Delta U = \frac{\Delta U}{\Delta F_n} \Delta F_n$$

Now we calculate the strain energy in a different manner. Let the elastic system be free of all forces. Let ΔF_n be applied first. The energy stored is

$$\frac{1}{2} \Delta F_n \, \Delta \delta_n$$

where $\Delta\delta_n$ is the elementary displacement corresponding to ΔF_n. This is a quantity of the second order which can be neglected since ΔF_n will be made to tend to zero in the limit. Next, we put all the other forces, F_1, F_2 , . . . ,etc. These forces by themselves do an amount of work equal to U. But while these displacements are taking place, the elementary force ΔF_n is acting all the time with full magnitude at the point n which is undergoing a displacement δ_n. Hence, this elementary force does work equal to $\Delta F_n \, \delta_n$. The total energy stored is therefore

$$U + \Delta F_n \, \delta_n + \frac{1}{2} \Delta F_n \, \Delta \delta_n$$

Equating this to the previous expression, we get

$$U + \frac{\Delta U}{\Delta F_n} \Delta F_n = U + \Delta F_n \, \delta_n + \frac{1}{2} \Delta F_n \, \Delta \delta_n$$

In the limit, when $\Delta F_n \to 0$

$$\frac{\partial U}{\partial F_n} = \delta_n$$

it is important to note that δ_n is a linear displacement if F_n is a concentrated load, or an angular displacement if F_n is a couple or a torque. Further, we must express the strain energy in terms of the forces (including moments and couples) since it is the partial derivative with respect to a particular force that gives the corresponding displacement. In the next section, expressions for strain energies in terms of forces will be obtained.

5.10 EXPRESSIONS FOR STRAIN ENERGY

In this section we shall develop expressions for strain energy when an elastic member is subjected to axial force, shear force, bending moment and torsion. Figure 5.10(a) shows an elastic member subjected to several forces. Consider a section of the member at C. In general, this section will be subjected to three forces F_x, F_y and F_z and three moments M_x, M_y and M_z (Fig. 5.10(b)). The force F_x is the axial force and forces F_y and F_z are the shear forces across the section. Moment M_x is the torque T and moments M_y and M_z are the bending moments about the y and z axes respectively. Let Δs be an elementary length of the member; then when Δs is very small, we can assume that these forces and moments remain constant over Δs. At the left-hand section of this elementary member, the forces and moments have opposite signs. During the deformation caused by the axial force F_x alone, the remaining forces and moments do no work. Similarly, during the twist caused by the torque $T = M_x$, no work is assumed to be done (since the deformations are extremely small) by the other forces and moments.

Consequently, the work done by each of these forces and moments can be determined individually and added together to determine the total elastic strain energy stored by Δs while it undergoes deformation. We shall make use of the formulas available from elementary strength of materials.

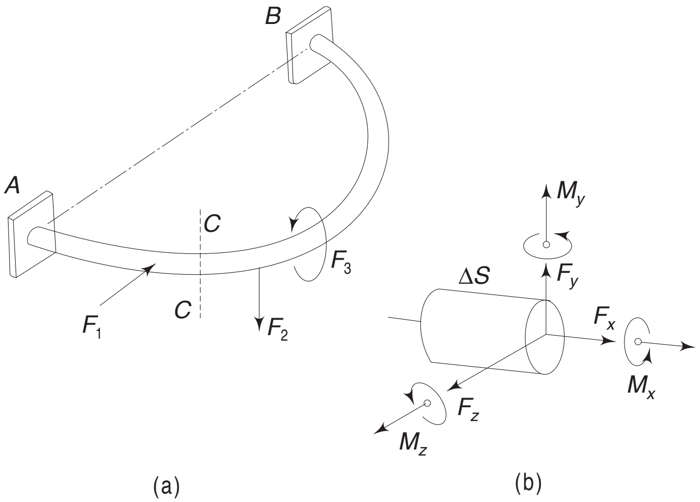

Fig. 5.10 *Reactive forces at a general cross-section*

(i) Elastic energy due to axial force: If δ_x is the axial extension due to F_x, then

$$\Delta U = \frac{1}{2} F_x \delta_x$$

$$= \frac{1}{2} F_x \cdot \frac{F_x}{AE} \Delta s$$

using Hooke's law.

$$\therefore \qquad \Delta U = \frac{F_x^2}{2AE} \Delta s \tag{5.17}$$

A is the cross-sectional area and E is Young's modulus.

(ii) Elastic energy due to shear force: The shear force F_y (or F_z) is distributed across the section in a complicated manner depending on the shape of the cross-section. If we assume that the shear force is distributed uniformly across the section (which is not strictly correct), the shear displacement will be (from Fig. 5.11) $\Delta s\ \Delta\gamma$ and the work done by F_y will be

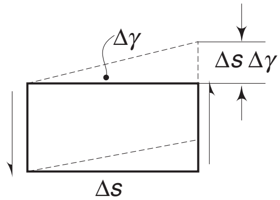

Fig. 5.11 *Displacement due to shear force*

$$\Delta U = \frac{1}{2} F_y\ \Delta s\ \Delta\gamma$$

From Hooke's law,

$$\Delta\gamma = \frac{F_y}{AG}$$

where A is the cross-sectional area and G is the shear modulus. Substituting this

$$\Delta U = \frac{1}{2} F_y\ \Delta s \frac{F_y}{AG}$$

or $$\Delta U = \frac{F_y^2}{2AG} \Delta s$$

It will be shown that the strain energy due to shear deformation is extremely small, which is often ignored. Hence, the error caused in assuming uniform distribution of the shear force across the section will be very small. However, to take into account the different cross-sections and non-uniform distribution, a factor k is introduced. With this

$$\Delta U = \frac{k F_y^2}{2AG} \Delta s \tag{5.18}$$

A similar expression is obtained for the shear force F_z.

(iii) Elastic energy due to bending moment: Making reference to Fig. 5.12, if $\Delta\phi$ is the angle of rotation due to the moment M_z(or M_y), the work done is

$$\Delta U = \frac{1}{2} M_z\ \Delta\phi$$

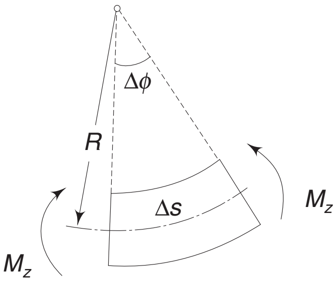

Fig. 5.12 *Displacement due to bending moment*

From the elementary flexure formula, we have

$$\frac{M_z}{I_z} = \frac{E}{R}$$

or
$$\frac{1}{R} = \frac{M_z}{EI_z}$$

where R is the radius of curvature and l_z is the area moment of inertia about the z axis. Hence

$$\Delta\phi = \frac{\Delta s}{R} = \frac{M_z}{EI_z}\Delta s$$

Substituting this

$$\Delta U = \frac{M_z^2}{2EI_z}\Delta s \tag{5.19}$$

A similar expression can be obtained for the moment M_y.

(iv) Elastic energy due to torque : Because of the torque T, the elementary member rotates through an angle $\Delta\theta$ according to the formula for a circular section

$$\frac{T}{I_p} = \frac{G\Delta\theta}{\Delta s}$$

i.e.
$$\Delta\theta = \frac{T}{GI_p}\Delta s$$

l_p is the polar moment of inertia. The work done due to this twist is,

$$\Delta U = \frac{1}{2}T\Delta\theta$$

$$= \frac{T^2}{2GI_p}\Delta s \tag{5.20}$$

Equations (5.17)–(5.20) give important expressions for the strain energy stored in the elementary length Δs of the elastic member. The elastic energy for the entire member is therefore

(i) Due to axial force
$$U_1 = \int_0^S \frac{F_x^2}{2AE}ds \tag{5.21}$$

(ii) Due to shear force
$$U_2 = \int_0^S \frac{k_y F_y^2}{2AG}ds \tag{5.22}$$

$$U_3 = \int_0^S \frac{k_z F_z^2}{2AG}ds \tag{5.23}$$

(iii) Due to bending moment
$$U_4 = \int_0^S \frac{M_y^2}{2EI_y}ds \tag{5.24}$$

$$U_5 = \int_0^S \frac{M_z^2}{2EI_z} ds \tag{5.25}$$

(iv) Due to torque

$$U_6 = \int_0^S \frac{T^2}{2GI_p} ds \tag{5.26}$$

Example 5.5 *Determine the deflection at end A of the cantilever beam shown in Fig. 5.13.*

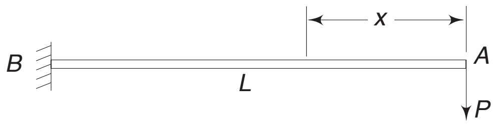

Fig. 5.13 *Example 5.5*

Solution The bending moment at any section x is

$$M = Px$$

The elastic energy due to bending moment is, therefore, from Eq. (5.24)

$$U_1 = \int_0^L \frac{(Px)^2 \, dx}{2EI} = \frac{P^2 L^3}{6EI}$$

The elastic energy due to shear from Eq. (5.22) is (putting $k_1 = 1$)

$$U_2 = \int_0^L \frac{P^2}{2AG} dx = \frac{P^2 L}{2AG}$$

One can now show that U_2 is small as compared to U_1. If the beam is of a rectangular section

$$A = bd, \quad I = \frac{1}{12} bd^3$$

and

$$2G \approx E$$

Substituting these

$$\frac{U_2}{U_1} = \frac{P^2 L}{2bdG} \cdot \frac{6bd^3}{12P^2 L^3} \cdot 2G$$

$$= \frac{d^2}{2L^2}$$

For a member to be designated as beam, the length must be fairly large compared to the cross-sectional dimension. Hence, $L > d$ and the above ratio is extremely small. Consequently, one can neglect shear energy as compared to bending energy. With

$$U = \frac{P^2 L^3}{6EI}$$

we get

$$\frac{\partial U}{\partial P} = \frac{PL^3}{3EI} = \delta_A$$

which agrees with the solution from elementary strength of materials.

Example 5.6 *For the cantilever of total length L shown in Fig. 5.14, determine the deflection at end A. Neglect shear energy.*

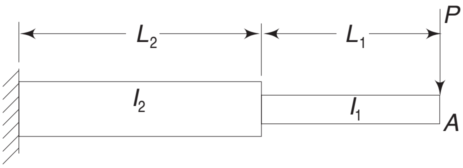

Fig. 5.14 *Example 5.6*

Solution The bending energy is

$$U = \int_0^{L_1} \frac{(Px)^2}{2EI_1}dx + \int_{L_1}^{L} \frac{(Px)^2}{2EI_2}dx$$

$$= \frac{P^2 L_1^3}{6EI_1} + \frac{P^2}{6EI_2}\left(L^3 - L_1^3\right)$$

$$\delta_A = \frac{\partial U}{\partial P} = \frac{P L_1^3}{3EI_1} + \frac{P}{3EI_2}\left(L^3 - L_1^3\right)$$

Example 5.7 *Determine the support reaction for the propped cantilever (Fig. 5.15.)*

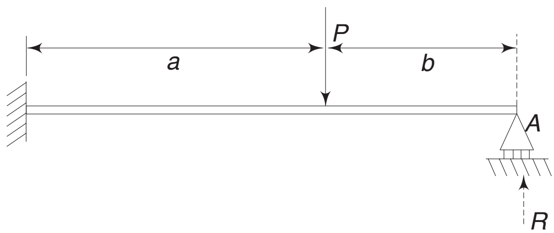

Fig. 5.15 *Example 5.7*

Solution The reaction R at A is such that the deflection there is zero. The energy is

$$U = \int_0^{b} \frac{(-Rx)^2}{2EI}dx + \int_0^{a} \frac{\left[-R(b+x)+Px\right]^2}{2EI}dx$$

$$U = \frac{1}{EI}\left(\frac{R^2 b^3}{6} + \frac{R^2 b^2 a}{2} + \frac{R^2 a^3}{6} + \frac{R^2 ba^2}{2}\right.$$

$$\left. + \frac{P^2 a^3}{6} - \frac{PRba^2}{2} - \frac{2PRa^3}{6}\right)$$

$$\frac{\partial U}{\partial R} = \frac{1}{EI}\left(\frac{Rb^3}{3} + Rb^2 a + \frac{Ra^3}{3} + Rba^2 - \frac{Pba^2}{2} - \frac{Pa^3}{3}\right)$$

Equating this to zero and solving for R,

$$R = \frac{Pa^2}{2}\,\frac{3b+2a}{(b+a)^3}$$

Remembering that $a + b = L$, the length of cantilever,

$$R = P\left(\frac{a}{L}\right)^2\left(\frac{3}{2} - \frac{a}{2L}\right)$$

Example 5.8 *For the structure shown in Fig. 5.16, what is the vertical deflection at end A?*

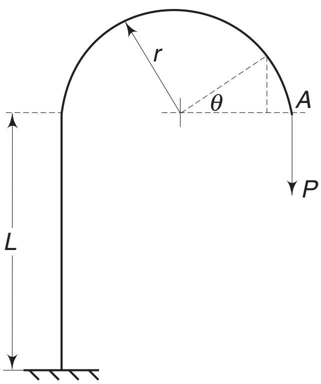

Fig. 5.16 *Example 5.8*

Solution The moment at any section θ of the curved part is $Pr\,(1 - \cos\theta)$. The bending moment for the vertical part of the structure is a constant equal to $2Pr$. The bending energy therefore is

$$\int_0^{\pi} \frac{\left[Pr\,(1-\cos\theta)\right]^2}{2EI}\, r\, d\theta + \int_0^{L} \frac{(2Pr)^2}{2EI}\, dx$$

We neglect the energy due to the axial force. Then

$$U = \frac{3}{4}\,\frac{\pi P^2 r^3}{EI} + \frac{2P^2 r^2 L}{EI}$$

$$\therefore\ \delta_A = \frac{\partial U}{\partial P} = \left(\frac{3}{2}\pi r + 4L\right)\frac{\mathrm{Pr}^2}{EI}$$

Example 5.9 *The end of the semi-circular member shown in Fig. 5.17, is subjected to torque T. What is the twist of end A? The member is circular in section.*

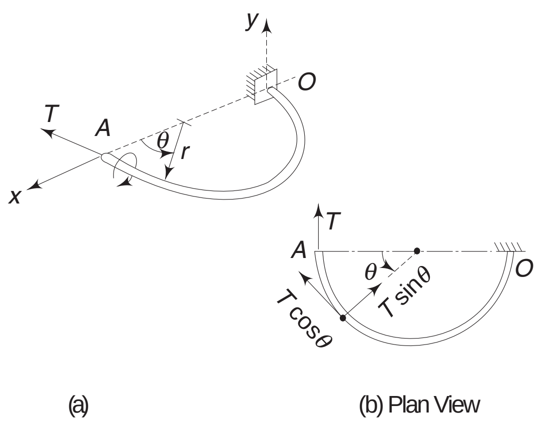

Fig. 5.17 *Example 5.9*

Solution The torque is a moment in the xy plane and can be represented by vector $\boldsymbol{T}$, as shown. At any section θ, this vector can be resolved into two components $\boldsymbol{T}\cos\theta$ and $\boldsymbol{T}\sin\theta$. The component $\boldsymbol{T}\cos\theta$ acts as torque and the component $\boldsymbol{T}\sin\theta$ as a moment.

The energy due to torque is, from Eq. (5.26),

$$U_1 = \int_0^{\pi} \frac{(T\cos\theta)^2}{2GI_P}\, r\, d\theta$$

$$= \frac{\pi r T^2}{4GI_P}$$

The energy due to bending is, from Eq. (5.24),

$$U_2 = \int_0^{\pi} \frac{(T\sin\theta)^2}{2EI}\, r\, d\theta$$

$$= \frac{\pi r T^2}{4EI}$$

I_p is the polar moment of inertia. For a circular member

$$I_p = 2I = \frac{\pi r^4}{2}$$

Substituting, the total energy is

$$U = U_1 + U_2 = \frac{\pi r T^2}{4}\left(\frac{1}{GI_p} + \frac{1}{EI}\right)$$

Hence, the twist is

$$\theta = \frac{\partial U}{\partial T} = \frac{\pi r T}{2}\left(\frac{1}{2G} + \frac{1}{E}\right)\frac{2}{\pi r^4}$$

$$= \frac{1}{r^3}\left(\frac{1}{2G} + \frac{1}{E}\right)T$$

5.11 FICTITIOUS LOAD METHOD

Castigliano's first theorem described above helps us to determine the displacement at a point corresponding to the force acting there. Situations arise where it may be desirable to determine the displacement (either linear or angular) at a point where there is no force (concentrated load or a couple) acting. In such situations, we assume a small fictitious or dummy load to be acting at the point where the displacement is required. Castigliano's theorem is then applied, and in the final result, the fictitious load is put equal to zero. The following example will describe the technique.

Example 5.10 *Determine the slope at end A of the cantilever in Fig. 5.18 which is subjected to load P.*

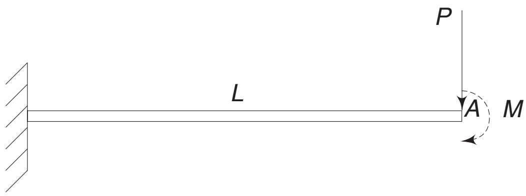

Fig. 5.18 *Example 5.10*

Solution To determine the slope by Castigliano's method we have to determine U and take its partial derivative with respect to the corresponding force, i.e. a moment. But no moment is acting at A. So, we assume a fictitious moment M to be acting at A and determine the slope caused by P and M. Since the magnitude of M is actually zero, in the final result, M is equated to zero.

The energy due to P and M is,

$$U = \int_0^L \frac{(Px + M)^2}{2EI}\,dx$$

$$= \frac{P^2 L^3}{6EI} + \frac{M^2 L}{2EI} + \frac{MPL^2}{2EI}$$

$$\theta = \frac{\partial U}{\partial M} = \frac{ML}{EI} + \frac{PL^2}{2EI}$$

This gives the slope when M and P are both acting. If M is zero, the slope due to P alone is

$$\theta = \frac{PL^2}{2EI}$$

If on the other hand, P is zero and M alone is acting the slope is

$$\theta = \frac{ML}{EI}$$

Example 5.11 *For the member shown in Fig. 5.16, Example 5.8, determine the ratio of L to r if the horizontal and vertical deflections of the loaded end A are equal. P is the only force acting.*

Solution In addition to the vertical for P at A, apply a horizontal fictitious force F to the right. The bending moment at section θ of the semi-circular part is

$$M_1 = Pr\,(1 - \cos\,\theta) - Fr\,\sin\,\theta)$$

At any section x in the vertical part, the moment is

$$M_2 = 2Pr + Fx$$

Hence,

$$U = \frac{1}{2EI}\int_0^{\pi}\left[Pr(1-\cos\theta) - Fr\,\sin\,\theta\right]^2 r\,d\theta + \frac{1}{2EI}\int_0^L (2Pr + Fx)^2\,dx$$

$$\therefore \quad \frac{\partial U}{\partial F} = -\frac{r^2}{EI}\int_0^{\pi}\left[\Pr(1-\cos\theta) - Fr\,\sin\theta\right]\sin\theta\,d\theta + \frac{1}{EI}\int_0^L (2Pr + Fx)\,x\,dx$$

and

$$\left.\frac{\partial U}{\partial F}\right|_{F=0} = \delta_h = -\frac{r^2}{EI}\int_0^{\pi}\left[Pr(1-\cos\theta)\sin\,\theta\right]d\theta + \frac{1}{EI}\int_0^L 2Pr\;xdx$$

$$= -\frac{2Pr^3}{EI} + \frac{PrL^2}{EI} = \frac{Pr}{EI}\left(-2r^2 + L^2\right)$$

From Example 5.8

$$\delta_v = \frac{Pr^2}{EI}\left(\frac{3}{2}\pi r + 4L\right)$$

Equating δ_v to δ_h

$$\frac{Pr^2}{EI}\left(\frac{3}{2}\pi r + 4L\right) = \frac{Pr}{EI}\left(-2r^2 + L^2\right)$$

or

$$L^2 - 4Lr - r^2\left(\frac{3\pi}{2} + 2\right) = 0$$

Dividing by r^2 and putting $\frac{L}{r} = \rho$

$$\rho^2 - 4\rho - \left(\frac{3\pi}{2} + 2\right) = 0$$

Solving, $$\rho = \frac{4 \pm \sqrt{[16 + 4(3\pi/2 + 2)]}}{2}$$

or $$\rho = 2 + \sqrt{6 + \frac{3}{2}\pi}$$

5.12 SUPERPOSITION OF ELASTIC ENERGIES

When an elastic body is subjected to several forces, one cannot obtain the total elastic energy by adding the energies caused by individual forces. In other words, the sum of individual energies is not equal to the total energy of the system. The reason for this is simple. Consider an elastic body subjected to two forces F_1 and F_2. When F_1 is applied first, let the energy stored be U_1. When F_2 is applied next (with F_1 continuing to act), the additional energy stored is equal to U_2 due to F_2 alone, plus the work done by F_1 during the displacement caused by F_2. Hence, the total energy stored when both F_1 and F_2 are acting is equal to ($U_1 + U_2 + U_3$), where U_1 is the work energy caused by F_1 alone, U_2 is the work energy caused by F_2 alone, and U_3 is the energy due to the work done by F_1 during the displacement caused by F_2. Another way of observing this is to note that the strain energy functions are not linear functions. Hence, individual energies cannot be added to get the total energy. As a specific example, consider the cantilever shown in Fig. 5.18, Example 5.10. Let P and M be actual forces acting on the cantilever, i.e M is not a fictitious force as was assumed in that example. The elastic energy stored due to P and M is given by (a), i.e.

$$U = \frac{P^2L^3}{6EI} + \frac{M^2L}{2EI} + \frac{MPL^2}{2EI}$$

The energy due to P alone is

$$U_1 = \frac{1}{2EI}\int_0^L (Px)^2\, dx = \frac{P^2L^3}{6EI}$$

Similarly, the energy due to M alone is

$$U_2 = \frac{1}{2EI}\int_0^L M^2\, dx = \frac{M^2L}{2EI}$$

Obviously, $U_1 + U_2$ is not equal to U. However, if P is applied first and then M, the total energy is given by $U_1 + U_2$ + work done by P during the displacement caused by M.

The deflection at the end of the cantilever (where P is acting with full magnitude) caused by M is

$$\delta_A^* = \frac{ML^2}{2EI}$$

During this deflection, the work done by P is

$$U_3 = P\left(\frac{ML^2}{2EI}\right)$$

If this additional energy is added to $U_1 + U_2$, then one gets the previous expression for U. It is immaterial whether P is applied first or M is applied. The order of loading is immaterial. Thus, one should be careful in applying the superposition principle to the energies. However, the individual energies caused by axial force, bending moment and torsion can be added since the force causing one kind of deformation will not do any work during a different kind of deformation caused by another force. For example, an axial force causing linear deformation will not do work during an angular deformation (or twist) caused by a torque. This is true in the case of small deformation as we have been assuming throughout our discussions. Similarly, a bending moment will not do any work during axial or linear displacement caused by an axial force.

5.13 STATICALLY INDETERMINATE STRUCTURE

Many statically indeterminate structural problems can be conveniently solved, using Castigliano's theorem. The technique is to determine the forces and moments to produce the required displacement. Example 5.7 was one such problem. The following example will further illustrate this method.

Example 5.12 *A rectangular frame with all four sides of equal cross section is subjected to forces P, as shown in Fig. 5.19. Determine the moment at section C and also the increase in the distance between the two points of application of force P.*

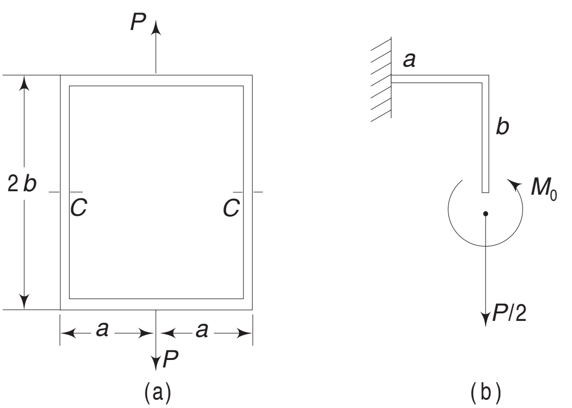

Fig. 5.19 *Example 5.12*

Solution The symmetry conditions indicate that the top and bottom members deform in such a manner that the tangents at the points of loading remain horizontal. Also, there is no change in slopes at sections C-C. Hence, one can consider only a quarter part of the frame, as shown in (b). Considering only the bending energy and neglecting the energies due to direct tension and shear force, we get

$$U' = \int_0^b \frac{M_0^2}{2EI}\,dx + \int_0^a \frac{\left(M_0 - P/2\,x\right)^2}{2EI}\,dx$$

$$= \frac{1}{2EI}\left(M_0^2\,b + M_0^2\,a - M_0 P\,\frac{a^2}{2} + \frac{1}{12}\,P^2 a^3\right)$$

Because of symmetry, the change in slope at section C is zero. Hence

$$\frac{\partial U'}{\partial M_0} = \frac{1}{2EI}\left[2M_0\,(a+b) - \frac{1}{2}\,Pa^2\right]$$

Equating this to zero,

$$M_0 = \frac{Pa^2}{4(a+b)}$$

To determine the increase in distance between the two load points, we determine the partial derivative of $4U'$ with respect to P (assuming that the bottom loaded point is held fixed).

$$U = 4U' = \frac{4}{2EI}\left[\frac{P^2a^4}{16(a+b)^2}(a+b) - \frac{P^2a^4}{8(a+b)} + \frac{P^2a^3}{12}\right]$$

$$\therefore \qquad \frac{\partial U}{\partial P} = \frac{Pa^3}{12EI}\frac{(a+4b)}{(a+b)}$$

Example 5.13 *A thin circular ring of radius r is subjected to two diametrically opposite loads P in its own plane as shown in Fig. 5.20(a). Obtain an expression for the bending moment at any section. Also, determine the change in the vertical diameter.*

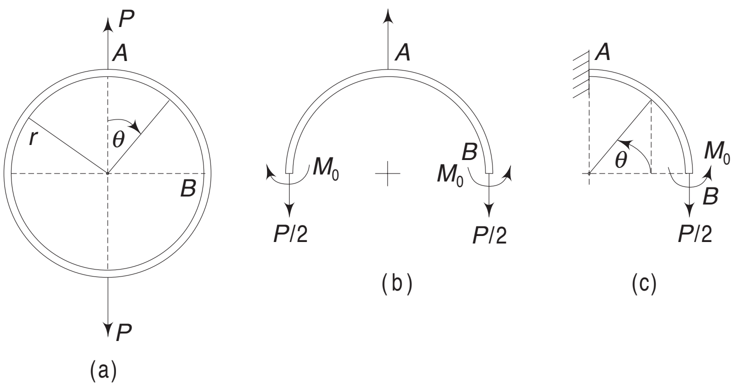

Fig. 5.20 *Example 5.13*

Solution Because of symmetry, during deformation there is no change in the slopes at A and B. So, one can consider only a quarter of the ring for calculation as shown in Fig. 5.20(c). The value of M_0 is such as to cause no change in slope at B. Section at A can be considered as built-in.

Moment at $\qquad \theta = M = \frac{P}{2}r(1-\cos\theta) - M_0$

$$U = \frac{1}{2EI}\int_0^{\pi/2}\left[\frac{P}{2}r(1-\cos\theta) - M_0\right]^2 r\,d\theta$$

Since there is no change in slope at B

$$\frac{\partial U}{\partial M_0} = -\frac{r}{2EI}\int_0^{\pi/2} 2\left[\frac{P}{2}r(1-\cos\theta) - M_0\right]d\theta = 0$$

i.e. $$\int_0^{\pi/2}\left[\frac{P}{2}r(1-\cos\theta)-M_0\right]d\theta=0$$

i.e. $$\frac{P}{2}r\left(\frac{\pi}{2}-1\right)-M_0\frac{\pi}{2}=0$$

or $$M_0=\frac{Pr}{2}\left(1-\frac{2}{\pi}\right)$$

$$\therefore \quad M \text{ at } \theta=\frac{P}{2}r(1-\cos\theta)-\frac{P}{2}r\left(1-\frac{2}{\pi}\right)=\frac{Pr}{2}\left(\frac{2}{\pi}-\cos\theta\right)$$

To determine the increase in the diameter along the loads, one has to determine the elastic energy and take the differential. If one considers the quarter ring, Fig. 5.20(c), the elastic energy is

$$U^*=\int_0^{\pi/2}\frac{1}{2EI}\left[\frac{Pr}{2}\left(\frac{2}{\pi}-\cos\theta\right)\right]^2 r\,d\theta$$

The differential of this with respect to ($P/2$) will give the vertical deflection of the end B with reference to A. Observe that in order to determine the deflection at B, one has to take the differential with respect to the particular load that is acting at that point, which is ($P/2$). Putting ($P/2$) = Q.

$$U^*=\frac{1}{2EI}\int_0^{\pi/2}\left[Qr\left(\frac{2}{\pi}-\cos\theta\right)\right]^2 r\,d\theta$$

$$=\frac{Q^2r^3}{2EI}\int_0^{\pi/2}\left(\frac{2}{\pi}-\cos\theta\right)^2 d\theta$$

$$\therefore \quad \frac{\partial U^*}{\partial Q}=\frac{Qr^3}{EI}\int_0^{\pi/2}\left(\frac{4}{\pi^2}+\cos^2\theta-\frac{4}{\pi}\cos\theta\right)d\theta$$

$$=\frac{Qr^3}{EI}\left(\frac{4}{\pi^2}\frac{\pi}{2}+\frac{\pi}{4}-\frac{4}{\pi}\right)$$

$$=\frac{Qr^3}{EI}\left(\frac{\pi}{4}-\frac{2}{\pi}\right)=\frac{Pr^3}{2EI}\left(\frac{\pi}{4}-\frac{2}{\pi}\right)$$

As this gives only the increase in the radius, the increase in the diameter is twice this quantity, i.e.

$$\delta_v=\frac{Pr^3}{EI}\left(\frac{\pi}{4}-\frac{2}{\pi}\right)$$

5.14 THEOREM OF VIRTUAL WORK

Consider an elastic system subjected to a number of forces (including moments) $F_1, F_2, \ldots,$ etc. Let $\delta_1, \delta_2, \ldots,$ etc. be the corresponding displacements. Remember that these are the work absorbing components (linear and angular displacements) in the corresponding directions of the forces (Fig. 5.21).

Let one of the displacements δ_1 be increased by a small quantity $\Delta\delta_1$. During this additional displacement, all other displacements where forces are acting are

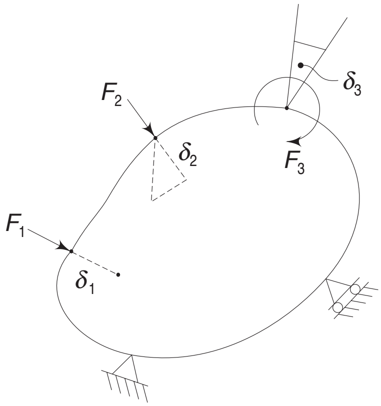

Fig. 5.21 *Generalised forces and displacements*

held fixed, which means that additional forces may be necessary to maintain such a condition. Further, the small displacement $\Delta\delta_1$ that is imposed must be consistent with the constraints acting. For example, if point I is constrained in such a manner that it can move only in a particular direction, then $\Delta\delta_1$ must be consistent with such a constraint. A hypothetical displacement of such a kind is called a virtual displacement. In applying this virtual displacement, the forces F_1, F_2, . . ., etc. (except F_1) do no work at all because their points of application do not move (at least in the work-absorbing direction). The only force doing work is F_1 by an amount $F_1\ \Delta\delta_1$ plus a fraction of $\Delta F_1\ \Delta\delta_1$, caused by the change in F_1. This additional work is stored as strain energy ΔU. Hence

$$\Delta U = F_1\ \Delta\delta_1 + k\ \Delta F_1\ \Delta\delta_1$$

or

$$\frac{\Delta U}{\Delta\delta_1} = F_1 + k\,\Delta F_1$$

and

$$\underset{\Delta\delta_1 \to 0}{\text{Lt}} \frac{\Delta U}{\Delta\delta_1} = \frac{\partial U}{\partial\delta_1} = F_1 \tag{5.27}$$

This is the theorem of virtual work. Note that in this case, the strain energy must be expressed in terms of δ_1, δ_2, . . ., etc. whereas in the application of Castigliano's theorem U had to be expressed in terms of F_1, F_2, . . ., etc.

It is important to observe that in obtaining the above equation, we have not assumed that the material is linearly elastic, i.e. that it obeys Hooke's law. The theorem is applicable to any elastic body, linear or nonlinear, whereas Castigliano's first theorem, as derived in Eq. (5.16), is strictly applicable to linear elastic or Hookean materials. This aspect will be discussed further in Sec. 5.15.

Example 5.14 *Three elastic members AD, BD and CD are connected by smooth pins, as shown in Fig. 5.22. All the members have the same cross-sectional areas and are of the same material. BD is 100 cm long and members AD and CD are each 200 cm long. What is the deflection of D under load W?*

Solution Under the action of load W, it is possible for D to move vertically and horizontally. If δ_1 and δ_2 are the vertical and horizontal displacements, then according to the principle of virtual work.

$$\frac{\partial U}{\partial\delta_1} = W, \quad \frac{\partial U}{\partial\delta_2} = 0$$

where U is the total strain energy of the system.

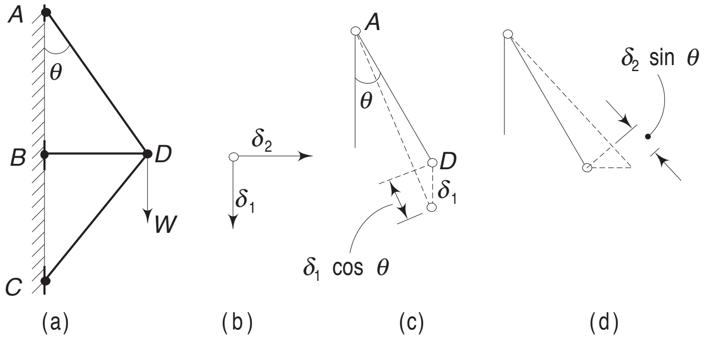

Fig. 5.22 *Example 5.14*

Becacse of δ_1, *BD* will not undergo any changes in length but *AD* will extend by $\delta_1 \cos\theta$ and *CD* will contract by the same amount, From Fig. (a),

$$\cos\ \theta = \frac{\sqrt{3}}{2}$$

Because of δ_2, *BD* will extend by δ_2 and *AD* and *CD* each will extend by $\frac{1}{2}\delta_2$. Hence, the total extension of each member is

$$AD \text{ extends by } \frac{1}{2}\left(\sqrt{3}\,\delta_1 + \delta_2\right) \text{ cm}$$

$$BD \text{ extends by } \delta_2 \text{ cm}$$

$$CD \text{ extends by } \frac{1}{2}\left(-\sqrt{3}\,\delta_1 + \delta_2\right) \text{ cm}$$

To calculate the strain energy, one needs to know the force-deformation equation for the non-Hookean members. This aspect will be taken up in Sec. 5.17, and Example 5.17. For the present example, assuming Hooke's law, the forces in the members are (with δ as corresponding extensions)

$$\text{in } AD: \frac{aE\delta}{L} = aE\,\frac{1}{2}\left(\sqrt{3}\,\delta_1 + \delta_2\right)\frac{1}{200}$$

$$\text{in } BD: \frac{aE\delta}{L} = aE\delta_2\,\frac{1}{100}$$

$$\text{in } CD: \frac{aE\delta}{L} = aE\,\frac{1}{2}\left(-\sqrt{3}\,\delta_1 + \delta_2\right)\frac{1}{200}$$

The total elastic strain energy taking only axial forces into account is

$$U = \Sigma\,\frac{P^2L}{2aE} = \frac{aE}{2}\left[\frac{1}{800}\left(\sqrt{3}\,\delta_1 + \delta_2\right)^2 + \frac{1}{100}\,\delta_2^2 + \frac{1}{800}\left(-\sqrt{3}\,\delta_1 + \delta_2\right)^2\right]$$

$$= aE\left(\frac{3}{800}\,\delta_1^2 + \frac{1}{160}\,\delta_2^2\right)$$

$$\therefore \qquad W = \frac{\partial U}{\partial \delta_1} = \frac{3aE}{400}\,\delta_1$$

and $$0 = \frac{\partial U}{\partial \delta_2} = \frac{aE}{80}\,\delta_2$$

Hence, δ_2 is zero, which means that D moves only vertically under W and the value of this vertical deflection δ_1 is

$$\delta_1 = \frac{400}{3aE}\,W$$

5.15 KIRCHHOFF'S THEOREM

In this section, we shall prove an important theorem dealing with the uniqueness of solution. First, we observe that the applied forces taken as a whole work on the body upon which they act. This means that some of the products $F_n\ \delta_n$ etc. may be negative but the sum of these products taken as a whole is positive. When the body is elastic, this work is stored as elastic strain energy. This amounts to the statement that U is an essentially positive quantity. If this were not so, it would have been possible to extract energy by applying an appropriate system of forces. Hence, every portion of the body must store positive energy or no energy at all. Accordingly, U will vanish only when every part of the body is undeformed. On the basis of this and the superposition principle, we can prove Kirchhoff's uniqueness theorem, which states the following:

An elastic body for which displacements are specified at some points and forces at others, will have a unique equilibrium configuration.

Let the specified displacements be $\delta_1, \delta_2, \ldots, \delta_r$ and the specified forces be F_s, $F_t, \ldots, F_n$. It is necessary to observe that it is not possible to prescribe simultaneously both force and displacement for one and the same point. Consequently, at those points where displacements are prescribed, the corresponding forces are $F_1', F_2', \ldots, F_r'$ and at those points where forces are prescribed, the corresponding displacement are $\delta_s', \delta_t', \ldots, \delta_n'$. Let this be the equilibrium configuration. If this system is not unique, then there should be another equilibrium configuration in which the forces corresponding to the displacements $\delta_1, \delta_2, \ldots, \delta_r$ have the values $F_1'', F_2'', \ldots, F_r''$ and the displacements corresponding to the forces $F_s, F_t, \ldots, F_n$ have the values $\delta_s'', \delta_t'', \ldots, \delta_n''$. We therefore have two distinct systems.

First System	Forces	$F_1', F_2', \ldots, F_r'$,	$F_s, F_t, \ldots, F_n$
	Corresponding displacements	$\delta_1, \delta_2, \ldots, \delta_r$	$\delta_s', \delta_t', \ldots, \delta_n'$
Second System	Forces	$F_1'', F_2'', \ldots, F_r''$	$F_s, F_t, \ldots, F_n$
	Corresponding displacements	$\delta_1, \delta_2, \ldots, \delta_r$	$\delta_s'', \delta_t'', \ldots, \delta_n''$

We have assumed that these are possible equilibrium configurations. Hence, by the principle of superposition the difference between these two systems must also be an equilibrium configuration. Subtracting the second system from the first, we get the third equilibrium configuration as

Forces	$(F_1' - F_1''), (F_2' - F_2''), \ldots, (F_r' - F_r'')$;	$0, \quad 0, \quad \ldots, \quad 0$
Corresponding displacements	$0, \quad 0 \quad \ldots, \quad 0$	$(\delta_s' - \delta_s''),\ (\delta_t' - \delta_t''),\ \ldots,\ (\delta_n' - \delta_n'')$

The strain energy corresponding to the third system is $U = 0$. Consequently the body remains completely undeformed. This means that the first and second systems are identical, i.e. there is a unique equilibrium configuration.

5.16 SECOND THEOREM OF CASTIGLIANO OR MENABREA'S THEOREM

This theorem is of great importance in the solution of redundant structures or frames. Let a framework consist of m number of members and j number of joints. Then, if

$$M > 3j - 6$$

the frame is termed a redundant frame. The reason is as follows. For each joint, we can write three force equilibrium equations (in a general three-dimensional case), thus giving a total of $3j$ number of equations. However, all these equation are not independent, since all the external forces by themselves are in equilibrium and, therefore, satisfy the three force equilibrium equations and the three moment equilibrium equations. Hence, the number of independent equations are $3j - 6$ and if the number of members exceed $3j - 6$, the frame is redundant. The number

$$N = m - 3j + 6$$

is termed the order of redundancy of the framework. If the skeleton diagram lies wholly in one plane, the framework is termed a plane frame. For a plane framework, the degree of redundancy is given by the number

$$N = m - 2j + 3$$

Castigliano's second theorem (also known as Menabrea's theorem) can be stated as follows:

The forces developed in a redundant framework are such that the total elastic strain energy is a minimum.

Thus, if F_1, F_2 and F_r are the forces in the redundant members of a framework and U is the elastic strain energy, then

$$\frac{\partial U}{\partial F_1} = 0, \quad \frac{\partial U}{\partial F_2} = 0, \ldots, \quad \frac{\partial U}{\partial F_r} = 0$$

This is also called the principle of least work and can be proven as follows:

Let r be the number of redundant members. Remove the latter and replace their actions by their respective forces, as shown in Fig. 5.23(b). Assuming that the values of these redundant forces $F_1, F_2, \ldots, F_r$ are known, the framework will have become statically determinate and the elastic strain energy of the remaining members can be determined. Let U_s be the strain energy of these members. Then by Castigliano's first theorem, the 'increase' in the distance between the joints a and b is given as

$$\delta'_{ab} = -\frac{\partial U_s}{\partial F_i} \tag{5.28}$$

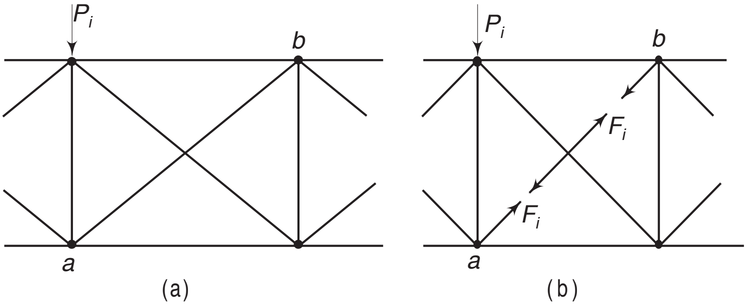

Fig. 5.23 *(a) Redundant structure (b) Structure with redundant member removed*

The negative appears because of the direction of F_i. The reactive force on the redundant members *ab* being F_i, its length will increase by

$$\delta_{ab} = \frac{F_i\, l_i}{A_i\, E_i} \tag{5.29}$$

where l_i is the length and A_i is the sectional area of the member. The increase in the distance given by Eq. (5.28) must be equal to the increase in the length of the member *ab*, given by Eq. (5.29). Hence

$$-\frac{\partial U_s}{\partial F_i} = \frac{F_i\, l_i}{A_i\, E_i} \tag{5.30}$$

The elastic strain energies of the redundant members are

$$U_1 = \frac{F_l^2\, l_1}{2A_1\, E_1},\quad U_2 = \frac{F_2^2\, l_2}{2A_2\, E_2}, \ldots,\quad U_r = \frac{F_r^2\, l_r}{2A_r\, E_r}$$

Hence, the total elastic energy of all redundant members is

$$U_1 + U_2 + \ldots U_r = \frac{F_l^2\, l_1}{2A_1\, E_1} + \frac{F_2^2\, l_2}{2A_2\, E_2} + \ldots + \frac{F_r^2\, l_r}{2A_r\, E_r}$$

$$\therefore \quad \frac{\partial}{\partial F_i}\left(U_1 + U_2 + \ldots + U_r\right) = \frac{F_i\, l_i}{A_i\, E_i}$$

since all terms, other than the *i*th term on the right-hand side, will vanish when differentiated with respect to F_i. Substituting this in Eq. (5.30)

$$-\frac{\partial U_s}{\partial F_i} = \frac{\partial}{\partial F_i}\left(U_1 + U_2 + \ldots + U_r\right) = 0$$

or $$\frac{\partial}{\partial F_i}\left(U_1 + U_2 + \ldots + U_r + U_s\right) = 0$$

The sum of the terms inside the parentheses is the total energy of the entire framework including the redundant members. If U is this total energy

$$\frac{\partial U}{\partial F_i} = 0$$

Similarly, by considering the redundant members one-by-one, we get

$$\frac{\partial U}{\partial F_1} = 0, \quad \frac{\partial U}{\partial F_2} = 0, \ldots, \frac{\partial U}{\partial F_r} = 0 \tag{5.31}$$

This is the principle of least work.

Example 5.15 *The framework shown in Fig. 5.24 contains a redundant bar. All the members are of the same section and material. Determine the force in the horizontal redundant member.*

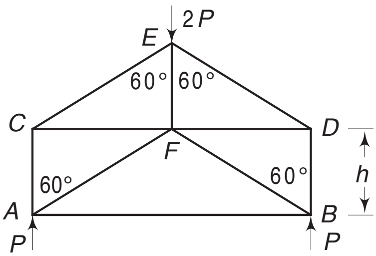

Fig. 5.24 *Example 5.15*

Solution Let T be the tension in the member AB. The forces in the members are

Members	*Length*	*Force*
AB	$2\sqrt{3}\,h$	$+T$
AC, BD	h	$T/\sqrt{3} - P$
AF, BF	$2h$	$-2T/\sqrt{3} + 0$
CF, DF	$\sqrt{3}\,h$	$-T + P\sqrt{3}$
CE, DE	$2h$	$2T/\sqrt{3} - 2P$
FE	h	$-2T/\sqrt{3} + 0$

The total strain energy is

$$U = \frac{h}{2EA}\left[2\sqrt{3}\,T^2 + 2\left(P^2 + \frac{T^2}{3} - \frac{2PT}{\sqrt{3}}\right) + \frac{16T^2}{3}\right.$$

$$+ 2\sqrt{3}\left(T^2 + 3P^2 - 2PT\sqrt{3}\right)$$

$$\left. + 16\left(\frac{T^2}{3} + P^2 - \frac{2PT}{\sqrt{3}}\right) + \frac{4T^2}{3}\right]$$

The condition for minimum strain energy or least work is

$$\frac{\partial U}{\partial T} = 0 = \frac{h}{2EA}\left[4\sqrt{3}T + \frac{4T}{3} - \frac{4P}{\sqrt{3}} + \frac{32T}{3} + 4\sqrt{3}\,T\right.$$

$$\left. -12P + \frac{32T}{3} - \frac{32}{\sqrt{3}}P + \frac{8T}{3}\right]$$

$$\therefore \qquad T\left(4\sqrt{3} + \frac{4}{3} + \frac{32}{3} + 4\sqrt{3} + \frac{32}{3} + \frac{8}{3}\right) = P\left(\frac{4}{\sqrt{3}} + 12 + \frac{32}{\sqrt{3}}\right)$$

or

$$T = \frac{9\left(\sqrt{3}+1\right)}{6\sqrt{3}+19}P$$

Example 5.16 *A cantilever is supported at the free end by an elastic spring of spring constant k. Determine the reaction at A (Fig. 5.25). The cantilever beam is uniformly loaded. The intensity of loading is W.*

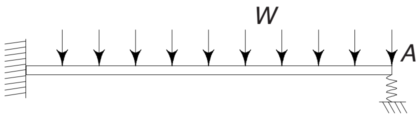

Fig. 5.25 *Example 5.16*

Solution Let R be the unknown reaction at A, i.e. R is the force on the spring. The strain energy in the spring is

$$U_1 = \frac{1}{2} R\delta = \frac{1}{2} R \frac{R}{k} = \frac{R^2}{2k}$$

where δ is the deflection of the spring. The strain energy in the beam is

$$U_2 = \int_0^L \frac{M^2\, dx}{2EI}$$

$$= \int_0^L \frac{\left(Rx - wx^2/2\right)^2 dx}{2EI}$$

$$= \frac{1}{EI}\left(\frac{1}{6} R^2 L^3 + \frac{1}{40} w^2 L^5 - \frac{1}{8} RwL^4\right)$$

Hence, the total strain energy for the system is

$$U = U_1 + U_2 = \frac{R^2}{2k} + \frac{1}{EI}\left(\frac{1}{6} R^2 L^3 + \frac{1}{40} w^2 L^5 - \frac{1}{8} RwL^4\right)$$

From Castigliano's second theorem

$$\frac{\partial U}{\partial R} = \frac{R}{k} + \frac{1}{EI}\left(\frac{1}{3} RL^3 - \frac{1}{8} wL^4\right) = 0$$

$$\therefore \qquad R = \frac{3kwL^4}{8\left(3EI + kL^3\right)}$$

5.17 GENERALISATION OF CASTIGLIANO'S THEOREM OR ENGESSER'S THEOREM

It is necessary to observe that in developing the first and second theorems of Castigliano, we have explicitly assumed that the elastic body satisfies Hooke's law, i.e. the body is linearly elastic. However, situations exist where the deformation is not proportional to load, though the body may be elastic. Consider the spring showns in Fig. 5.26(a), whose load–displacement curve is as given in Fig. 5.26(b).

The spring is a non-linear spring. Consider the area of OBC which is the strain energy. It is represented by

$$U = \int_0^x F\, dx \tag{5.32}$$

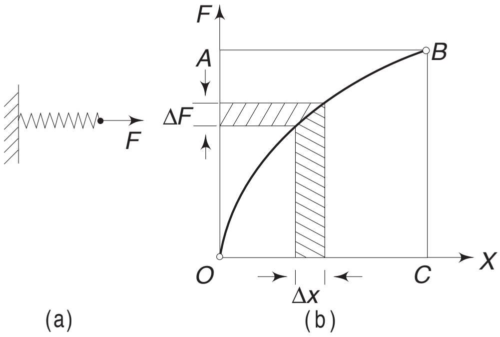

Fig. 5.26 *(a) Non-linear spring; (b) Non-linear load-displacement curve*

Hence $\dfrac{dU}{dx} = F$

This is the principle of virtual work, discussed in Sec. 5.14, and is applicable whether the elastic member is linear or non-linear. Now consider the area *OAB*. It is represented by

$$U^* = \int_0^F x\, dF \tag{5.33}$$

This is termed as a complementary energy. Differentiating the complementary energy with respect to *F* yields

$$\frac{dU^*}{dF} = x \tag{5.34}$$

This gives the deflection in the direction of *F*. If we compare with Castigliano's first theorem (Eq. 5.16), we notice that to obtain the corresponding deflection, we must take the derivative of the complementary energy and not that of the strain energy. When a material obeys Hooke's law, the curve *OB* is a straight line and consequently, the strain energy and the complementary strain energy are equal and it becomes immaterial which one we use in Castigliano's first theorem. The expression given by Eq. (5.34) represents Engesser's theorem.

Consider as an example an elastic spring the force deflection characteristic of which is represented by

$$F = ax^n$$

where a and n are constants.

The strain energy is

$$U = \int_0^x F\, dx = \int_0^x a(x')^n\, dx' = \frac{1}{n+1} a x^{n+1}$$

The complimentary strain energy is

$$U^* = \int_0^F x\, dF = \int_0^F \left(\frac{F}{a}\right)^{1/n} dF$$

$$= \frac{1}{a^{1/n}} \cdot \frac{n}{n+1} F^{(1+1/n)}$$

From these $\dfrac{dU}{dx} = ax^n = F$

$$\frac{dU^*}{dF} = \frac{1}{a^{1/n}} \cdot F^{1/n} = x$$

Further, expressing U in terms of F, we get

$$U = \frac{1}{n+1} \cdot a \left[\frac{1}{a^{1/n}} \cdot F^{1/n}\right]^{n+1}$$

$$\therefore \qquad \frac{dU}{dF} = \frac{1}{n}\left(\frac{F}{a}\right)^{1/n} = \frac{1}{n}x$$

and this does not agree with the correct result. Hence the principle of virtual work is valid both for linear and non-linear elastic material, whereas to obtain deflection using Castigliano's first theorem, we have to use the complementary energy U^* if the material is non-linear. If it is linearly elastic, it is immaterial wheather we use U or U^*, since both are equal.

Example 5.17 *Consider Fig. 5.27, which shows two identical bars hinged together, carrying a load W. Check Castigliano's first theorem, using the elastic and complementary strain energy.*

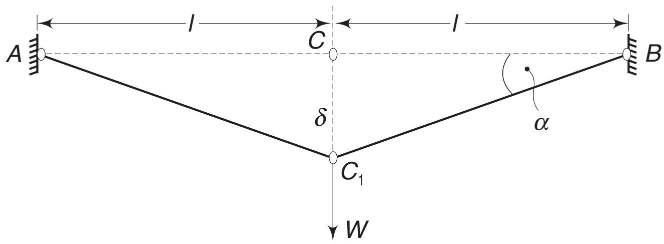

Fig. 5.27 *Example 5.17*

Solution When C has displacement $CC_1 = \delta$, we have from the figure for small α,

$$\tan\alpha \approx \sin\alpha \approx \delta/l$$

If F is the force in each member, a the cross-sectional area and ε the strain, then

$$F = \frac{W}{2\sin\alpha} \approx \frac{Wl}{2\delta}$$

and $$\varepsilon = \frac{\sqrt{l^2+\delta^2} - l}{l} \approx \frac{1}{2}\frac{\delta^2}{l^2}$$

Also $$\varepsilon = \frac{F}{aE} = \frac{Wl}{2\delta aE}$$

Equating the two strains

$$\frac{Wl}{2\delta aE} = \frac{\delta^2}{2l^2}$$

or $$\delta = l\left(\frac{W}{Ea}\right)^{1/3}$$

i.e. the deflection is not linearly related to the load.

The strain energy is

$$U = \int_0^{\delta} W\,d\delta = \frac{lW^{4/3}}{(aE)^{1/3}}$$

$$\therefore \qquad \frac{\partial U}{\partial W} = \frac{4lW^{1/3}}{3(aE)^{1/3}}$$

Hence, Castigliano's first theorem applied to the strain energy, does not yield the deflection δ. This is so because the load defection equation is not linearly related. If we consider the complementary energy,

$$U^* = \int_0^W \delta\, dW = \frac{l}{(Ea)^{1/3}} \int_0^W W^{1/3}\, dW$$

$$= \frac{3lW^{4/3}}{4(Ea)^{1/3}}$$

$$\frac{\partial U^*}{\partial W} = l\left(\frac{W}{Ea}\right)^{1/3} = \delta$$

Hence, Engesser's theorem gives the correct result.

5.18 MAXWELL–MOHR INTEGRALS

Castigliano's first theorem gives the displacement of points in the directions of the external forces where they are acting. When a displacement is required at a point where no external force is acting, a fictitious force in the direction of the required displacement is assumed at the point, and in the final result, the value of the fictitious load is considered equal to zero. This technique was discussed in Sec. 5.11. In this section, we shall develop certain integrals, which are based on the fictitious load techniques.

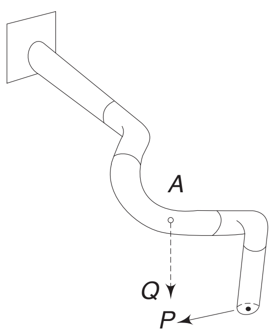

Fig. 5.28 *A general structure under load P*

Consider the determination of the vertical displacement of point A of a structure which is loaded by a force P, as shown in Fig. 5.28. Since no external force is acting at A in the corresponding direction, we apply a fictitious force Q in the corresponding direction at A. In order to calculate the strain energy in the elastic member, we need to determine the moments and forces across a general section. This is shown in Fig. 5.29.

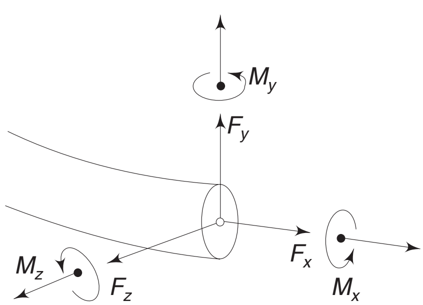

Fig. 5.29 *Moments and forces across a general section*

At any section, the moments and forces of reaction are caused by the actual external forces plus the fictitious load Q. For example, about the x axis we have

$$F_X = F_{xP} + F_{xQ},$$
$$M_X = M_{xP} + M_{xQ}$$

where F_{xP} is caused by the actual external forces, such as P, and F_{xQ} is due to the fictitious load Q. It is essential to observe that the additional force factors, such as F_{xQ}, M_{xQ}, etc. are directly proportional to Q. If Q is doubled, these factors also get doubled. Hence, one can write these as $F_{x1}Q$, $M_{x1}Q$, etc. where F_{x1}, M_{x1}, etc. are the force factors caused by a unit fictitious generalised force. Consequently, the force factors due to the actual loads and fictitious force are

$$\begin{aligned} F_x &= F_{xP} + F_{x1}Q, \qquad & M_x &= M_{xP} + M_{x1}Q \\ F_y &= F_{yP} + F_{y1}Q, \qquad & M_y &= M_{yP} + M_{y1}Q \\ F_z &= F_{zP} + F_{z1}Q, \qquad & M_z &= M_{zP} + M_{z1}Q \end{aligned} \tag{5.35}$$

Note that in Fig. 5.29 while M_x acts as a torque, M_y and M_z act as bending moments. These force factors vary from section to section. The total elastic energy is

$$\begin{aligned} U = {} & \int_l \frac{(M_{xP} + M_{x1}Q)^2\, ds}{2GI_x} + \int_l \frac{(M_{yP} + M_{y1}Q)^2\, ds}{2EI_y} \\ & + \int_l \frac{(M_{zP} + M_{z1}Q)^2\, ds}{2EI_z} + \int_l \frac{(F_{xP} + F_{x1}Q)^2\, ds}{2EA} \\ & + \int_l \frac{k_y (F_{yP} + F_{y1}Q)^2\, ds}{2GA} + \int_l \frac{k_z (F_{zP} + F_{z1}Q)^2\, ds}{2GA} \end{aligned}$$

Differentiating the above expression with respect to Q and putting $Q = 0$

$$\begin{aligned} \delta_A = \left.\frac{\partial U}{\partial Q}\right|_{Q=0} = {} & \int_l \frac{M_{xP}\, M_{x1}\, ds}{GI_x} + \int_l \frac{M_{yP}\, M_{y1}\, ds}{EI_y} \\ & + \int_l \frac{M_{zP}\, M_{z1}\, ds}{EI_z} + \int_l \frac{F_{xP}\, F_{x1}\, ds}{EA} \\ & + \int_l \frac{k_y\, F_{yP}\, F_{y1}\, ds}{GA} + \int_l \frac{k_z\, F_{zP}\, F_{z1}\, ds}{GA} \end{aligned} \tag{5.36}$$

If the fictitious force Q is replaced by a fictitious moment or torque, we get the corresponding deflection θ_A.

These sets of integrals are known as Maxwell–Mohr integrals. The above method is sometimes known as the unit load method. These integrals can be used to solve not only problems of finding displacements but also to solve problems connected with plane thin-walled rings. The above set of equations is generally written as

$$\begin{aligned} \delta_A = {} & \int_l \frac{M_x \bar{M}_x}{GI_x}\, ds + \int_l \frac{M_y \bar{M}_y}{EI_y}\, ds + \int_l \frac{M_z \bar{M}_z}{EI_z}\, ds \\ & + \int_l \frac{F_x \bar{F}_x}{EA}\, ds + \int_l \frac{k_y F_y \bar{F}_y}{GA}\, ds + \int_l \frac{k_z F_z \bar{F}_z}{GA}\, ds \end{aligned} \tag{5.37}$$

where $\overline{M}_x$, $\overline{M}_y$,. . . , $\overline{M}_z$ are the force factors caused by a generalised unit fictitious force applied where the appropriate displacement is needed.

Example 5.18 *Determine by what amount the straight portions of the ring are bought closer together when it is loaded, as shown in Fig. 5.30 consider only the bending energy.*

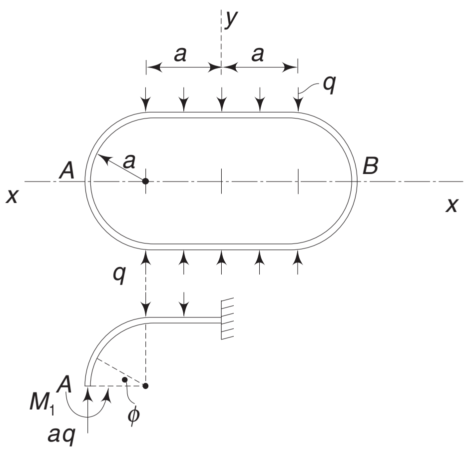

Fig. 5.30 *Example 5.18*

Solution Consider one quarter of the ring. The unknown moment M_1 is the redundant unknown generalised force. Owing to symmetry, there is no rotation of the section at point A. To determine the rotation, we assume a unit moment in the same direction as M_1. The moment due to this fictitious unit moment at any section is $\overline{M}$.

M at any section in quadrant = $aq\,.\,a\,(1 - \cos\phi) - M_1$

$\overline{M}$ at any section in quadrant = -1

M at any section in the top horizontal member = $aq\ \ (a + x) - qx^2/2 - M_1$

$\overline{M}$ at any section in the top horizontal member = -1

$$\therefore \qquad \theta_A = \int_0^{\pi/2} \frac{-a^2 q(1-\cos\phi) + M_1}{EI}\, a\, d\phi - \int_0^a \frac{a\,q(a+x) - qx^2/2 - M_1}{EI}\, dx$$

or
$$EI\theta_A = -a^3 q\left(\frac{\pi}{2} + \frac{1}{3}\right) + M_1 a\left(\frac{\pi}{2} + 1\right) = 0$$

$$\therefore \qquad M_1 = a^2 q \frac{3\pi + 2}{3(\pi + 2)} \approx 0.74\, a^2 q$$

This is the value of the redundant unknown moment. To determine the vertical displacements of the midpoints of the horizontal members, we apply a fictitious force $P_f = 1$ in an upward direction at point A of the quarter ring. Because of this

$\overline{M}$ at any section in quadrant = $-a\,(1 - \cos\phi)$

$\overline{M}$ at any section in top horizontal part = $-(a + x)$

Hence, the vertically upward displacement of point A is

$$\delta_A = \int_0^{\pi/2} -\frac{a^4 q(1-\cos\phi - 0.74)(1-\cos\phi)}{EI}\, d\phi$$

$$+ \int_0^a -\frac{\left[aq\,(a+x) - \frac{1}{2}qx^2 - 0.74a^2 q\right](a+x)}{EI}\, dx$$

$$= \frac{0.86 a^4 q}{EI}$$

Hence, the two horizontal members approach each other by a distance equal to

$$\frac{2\,(0.86)\,a^4 q}{EI} = 1.72\,\frac{a^4 q}{EI}$$

Example 5.19 *A thin walled circular ring is loaded as shown in Fig. 5.31. Determine the vertical displacement of point A. Take only the bending energy.*

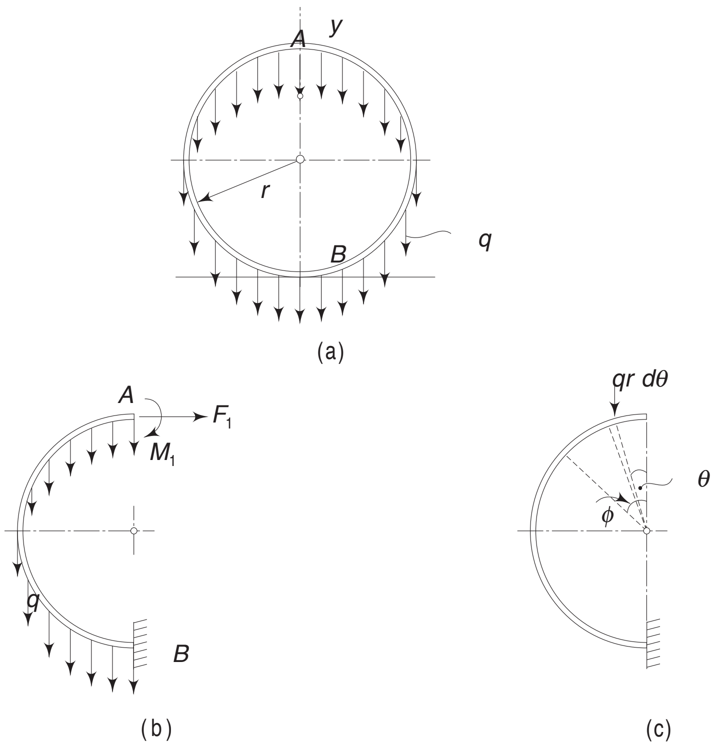

Fig. 5.31 *Example 5.19*

Solution Because of symmetry, we may consider one half of the ring. The reactive forces at section A are F_1 and M_1. Because of symmetry, section A does not rotate and also does not have a horizontal displacement. Hence in addition to M_1 and F_1, we assume a fictitious moment and a fictitious horizontal force, each of unit magnitude at section A.

The moment at any section ϕ due to the distributed loading q is

$$M_q = \int_0^{\phi} qr\,d\theta\,r\,(\sin\phi - \sin\theta) = qr^2(\phi\sin\phi + \cos\phi - 1)$$

M at any section ϕ with distributed loading F_1 and M_1 is

$$M = qr^2\,(\phi\,\sin\phi + \cos\phi - 1) + M_1 + F_1 r\,(1 - \cos\phi)$$

$\bar{M}$ at any section ϕ due to the unit fictitious horizontal force is

$$\bar{M} = r(1 - \cos\phi)$$

$$\therefore \quad \delta_A = \frac{I}{EI}\int_0^{\pi} r^2\left[qr^2\left(\phi\sin\phi + \cos\phi - 1\right) + M_1 + F_1 r\left(1-\cos\phi\right)\right]\left(1-\cos\phi\right)d\phi$$

$$= \frac{r^2}{EI}\left(-qr^2\frac{\pi}{4} + \pi M_1 + F_1 r\frac{3\pi}{2}\right)$$

Since this is equal to zero, we have

$$M_1 + \frac{3}{2}F_1 r = \frac{1}{4}qr^2 \tag{5.38}$$

$\bar{M}$ at any section ϕ due to unit fictitious moment is

$$\bar{M} = 1$$

$$\therefore \quad \theta_A = \frac{I}{EI}\int_0^{\pi} r\left[qr^2\left(\phi\sin\phi + \cos\phi - 1\right) + M_1 + F_1 r\left(1-\cos\phi\right)\right]d\phi$$

$$= \frac{r}{EI}\left(\pi M_1 + F_1 r\pi\right)$$

Since this is also equal to zero, we have

$$M_1 + F_1 r = 0 \tag{5.39}$$

Solving Eqs (5.38) and (5.39)

$$M_1 = -\frac{qr^2}{2} \quad \text{and} \quad F_1 = \frac{qr}{2}$$

To determine the vertical displacement of A we apply a fictitious unit force $P_f = 1$ at A in the downward direction.

$\bar{M}$ at any section ϕ due to $P_f = 1$ is $r\sin\phi$

$$\therefore \quad \delta_v = \int_0^{\pi} r^2\left[qr^2\left(\phi\sin\phi + \cos\phi - 1\right) + M_1 + F_1 r\left(1-\cos\phi\right)\right]\sin\phi\, d\phi$$

$$= \left(\frac{\pi^2}{4} - 2\right)\frac{qr^4}{EI} \approx 0.467\frac{qr^4}{EI}$$

Example 5.20 *Figure 5.32 shows a circular member in its plan view. It carries a vertical load W at A perpendicular to the plane of the paper. Taking only bendng and torsional energies into account, determine the vertical deflection of the loaded end A. The radius of the member is R and the member subtends an angle α at the centre.*

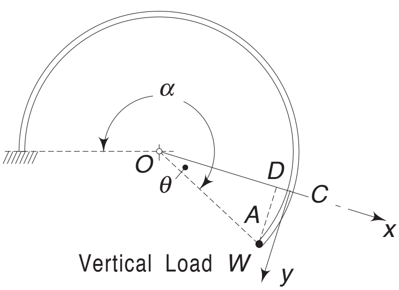

Fig. 5.32 *Example 5.20*

Solution At section C, the moment of the force about x axis acts as bendng moment M and the moment about y axis acts as torque T. Hence,

$$M = W \times AD = WR \sin\theta$$
$$T = W \times DC = WR\,(1 - \cos\theta)$$

$$\therefore \qquad U = \int_0^{\alpha} \frac{1}{2EI}(WR \sin\theta)^2\, R\, d\theta + \int_0^{\alpha} \frac{1}{2GJ}\left[WR(1-\cos\theta)\right]^2 R\, d\theta$$

When the load W is gradually applied, the work done by W during its vertical deflection is $\frac{1}{2}\,W\,\delta_V$ and this is stored as the elastic energy U. Thus,

$$\frac{1}{2}W\delta_V = U = \int_0^{\alpha} \frac{1}{2EI}(WR \sin\theta)^2\, R\, d\theta + \int_0^{\alpha} \frac{1}{2GJ}\left[WR(1-\cos\theta)\right]^2 R\, d\theta$$

or $$\delta_V = WR^3\left[\frac{1}{2EI}\left(\alpha - \frac{1}{2}\sin 2\alpha\right) + \frac{1}{GJ}\left(\frac{3}{2}\alpha + \frac{1}{4}\sin 2\alpha - 2\sin\alpha\right)\right]$$

This is the same as $\partial U/\partial W$.

if $$\alpha = \frac{\pi}{2}, \quad \delta_V = WR^3\left[\frac{\pi}{4EI} - \frac{1}{GJ}\left(\frac{3\pi - 8}{4}\right)\right]$$

if $$\alpha = \pi, \quad \delta_V = WR^3\,\pi\left(\frac{1}{EI} - \frac{3}{GJ}\right)$$

Problems

5.1 A load $P = 6000$ N acting at point R of a beam shown in Fig. 5.33 produces vertical deflections at three points A, B, and C of the beam as

$$\delta_A = 3 \text{ cm} \qquad \delta_B = 8 \text{ cm} \qquad \delta_C = 5 \text{ cm}$$

Find the deflection of point R when the beam is loaded at points, A, B and C by

$$P_A = 7500 \text{ N},\ P_B = 3500 \text{ N and } P_C = 5000 \text{ N}.$$

[*Ans.* 12.6 cm (approx.)]

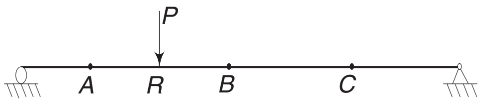

Fig. 5.33 *Problem 5.1*

5.2 For the horizontal beam shown in Fig. 5.34, a vertical displacement of 0.6 cm of support B causes a reaction $R_a = 10{,}000$ N at A. Determine the reaction R_b at B due to a vertical displacement of 0.8 cm at support A. [*Ans.* $R_b = 13{,}333$ N]

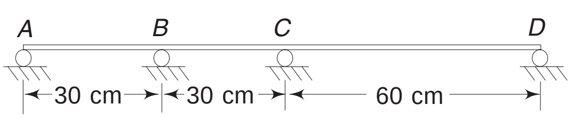

Fig. 5.34 *Problem 5.2*

5.3 A closed circular ring made of inextensible material is subjected to an arbitrary system of forces in its plane. Show that the area enclosed by the frame does not change under this loading. Assume small displacements (Fig. 5.35).

[*Hint:* Subject the ring to uniform internal pressure. Since the material is inextensible, no deformation occurs. Now apply the reciprocal theorem.]

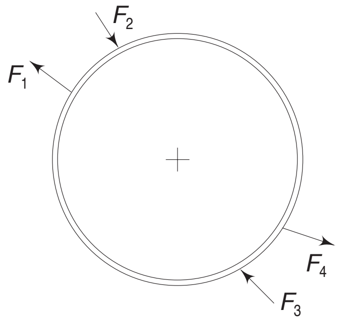

Fig. 5.35 *Problem 5.3*

5.4 Determine the vertical displacement of point A for the structure shown in Fig. 5.36. All members have the same cross-section and the same rigidity EA.

$$\left[Ans.\ \delta_A = \frac{Wl}{EA}\left(7 + 4\sqrt{2}\right)\right]$$

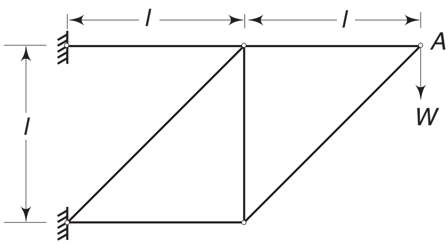

Fig. 5.36 *Problem 5.4*

5.5 Determine the rotation of point C of the beam under the action of a couple M applied at its centre (Fig. 5.37).

$$\left[Ans.\ \theta = \frac{M\,l}{12EI}\right]$$

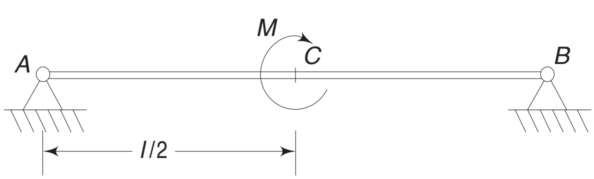

Fig. 5.37 *Problem 5.5*

5.6 What is the relative displacement of points *A* and *B* in the framework shown? Consider only bending energy (Fig. 5.38).

$$\left[Ans.\ \delta_{AB} = \frac{Pa^3}{6EI} + \frac{Pa^2 b}{2EI} \right]$$

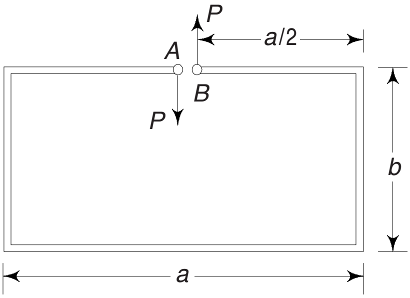

Fig. 5.38 *Problem 5.6*

5.7 What is the relative displacement of points *A* and *B* when subjected to forces *P*. Consider only bending energy (Fig. 5.39).

$$\left[Ans.\ \delta_{AB} = 3\pi \frac{PR^3}{EI} \right]$$

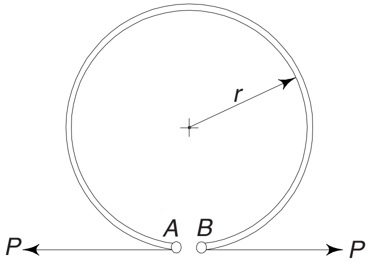

Fig. 5.39 *Problem 5.7*

5.8 Determine the vertical displacement of the point of application of force *P*. Take all energies into account. The member is of uniform circular cross-section (Fig. 5.40).

$$\left[Ans.\ \ \delta_A = 2P\left(\frac{a^3}{3EI} + \frac{a^2 b}{2EI} + \frac{a^3}{2GI_P} + \frac{ka}{AG} + \frac{b}{2AE} \right) \right]$$

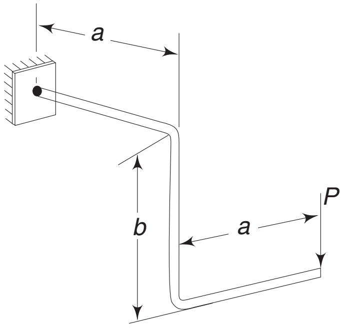

Fig. 5.40 *Problem 5.8*

5.9 What are the horizontal and vertical displacements of point A in Fig. 5.41. Assume AB to be rigid.

$$\left[Ans. \quad \delta_V = \frac{17Ph}{EA}; \delta_H = \frac{1.73Ph}{EA}\right]$$

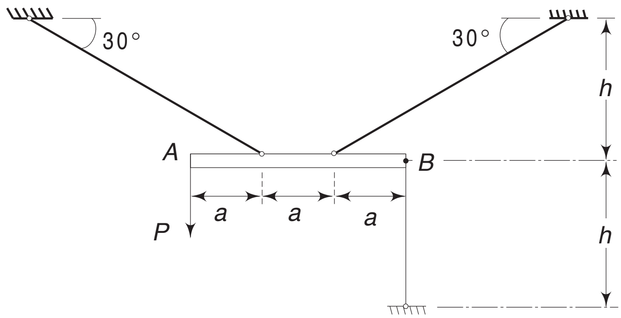

Fig. 5.41 *Problem 5.9*

5.10 Determine the vertical displacement of point B under the action of W. End B is free to rotate but can move only in a vertical direction (Fig. 5.42).

$$\left[Ans. \quad \delta_B = \frac{Wa^3}{EI}\left(\frac{3\pi}{4} - \frac{1}{9\pi + 8}\right)\right]$$

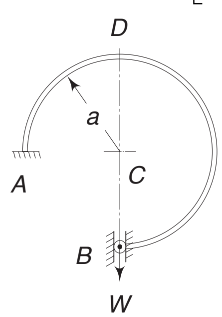

Fig. 5.42 *Problem 5.10*

5.11 Two conditions must be satisfied by an ideal piston ring. (a) It should be truly circular when in the cylinder, and (b) it should exert a uniform pressure all around. Assuming that these conditions are satisfied by specifying the initial shape and the cross-section, show that the initial gap width must be $3\pi pr^4/EI$, if the ring is closed inside the cylinder. p is the uniform pressure per centimetre of circumference. EI is kept constant.

5.12 For the torque measuring device shown in Fig. 5.43 determine the stiffness of the system, i.e. the torque per unit angle of twist of the shaft. Each of the springs has a length l and moment of inertia I for bending in the plane of the moment.

$$\left[Ans. \quad \frac{M}{\theta} \approx \frac{8EI}{l}\right]$$

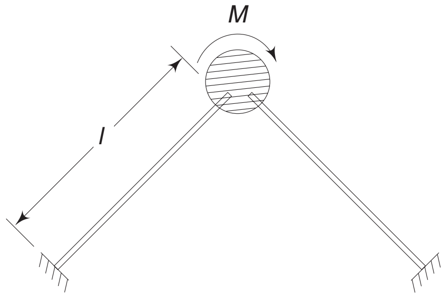

Fig. 5.43 *Problem 5.12*

5.13 A circular steel hoop of square cross-section is used as the controlling element of a high speed governor (Fig. 5.44). Show that the vertical deflection caused by angular velocity ω is given by

$$\delta = \frac{2\rho}{E}\frac{\omega^2 r^5}{t^2}$$

where r is the hoop radius, t the thickness of the section and ρ the weight density of the material.

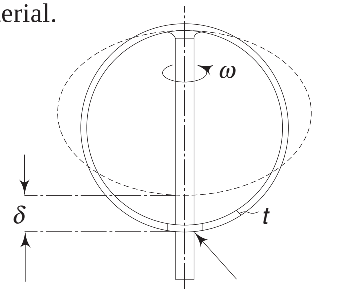

Fig. 5.44 *Problem 5.13*

5.14 A thin circular ring is loaded by three forces P as shown in Fig. 5.45. Determine the changes in the radius of the ring along the line of action of the forces. The included angle between any two forces is 2α and A is the cross-sectional area of the member. Consider both bending and axial energies.

$$\left[\textit{Ans.}\ \frac{PR^3}{2EI}\left(\cot\frac{\alpha}{2} + \frac{\alpha}{2\sin^2\alpha} - \frac{1}{\alpha}\right) + \frac{PR}{4EA}\left(\cot\alpha + \frac{\alpha}{\sin^2\alpha}\right)\right]$$

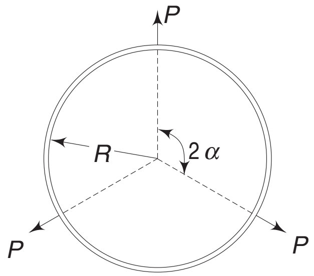

Fig. 5.45 *Problem 5.14*

5.15 For the system shown (Fig. 5.46) determine the load W necessary to cause a displacement δ in the vertical direction of point O. a is the cross-sectional area of each member and l is the length of each member. Use the principle of virtual work.

$$\left[Ans.\ W = \frac{3aE}{2l}\,\delta \right]$$

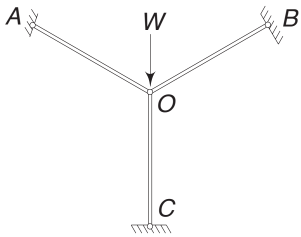

Fig. 5.46 *Problem 5.15*

5.16 In the previous problem determine the force in the member OC by Castigliano's second theorem. [*Ans.* 2W/3]

5.17 Using Castigliano's second theorem, determine the reaction of the vertical support C of the structure shown (Fig. 5.47). Beam ACB has Young's modulus E and member CD has a value E'. The cross-sectional area of CD is a.

$$\left[Ans. \frac{5wl^4aE'}{4\left(6EIh + qE'\,l^3\right)} \right]$$

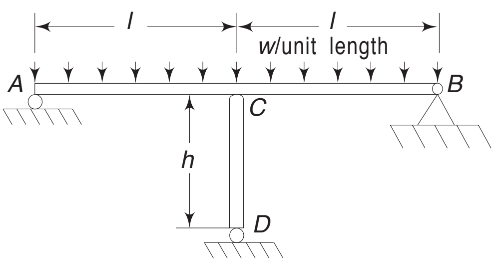

Fig. 5.47 *Problem 5.17*

5.18 A pin jointed framework is supported at A and D and it carries equal loads W at E and F. The lengths of the members are as follows:

$AE = EF = FD = BC = a$

$BE = CF = h$

$BF = CE = AB = CD = l = (a^2 + h^2)^{1/2}$

The cross-sectional areas of BF and CE are A_1 each, and of all the other members are A_2 each. Determine the tensions in BF and CE.

$$\left[Ans. \frac{WA_1\, lh^2}{A_1\left(a^3 + h^3\right) + A_2 l^3} \right]$$

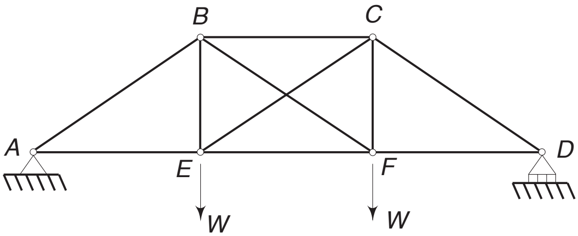

Fig 5.48 *Problem 5.18*

5.19 A ring is made up of two semi-circles of radius a and of two straight lines of length $2a$, as shown in Fig. 5.49. When loaded as shown, determine the change in distance between A and B. Consider only bending energy.

$$\left[\textit{Ans.} \frac{6 - 17\pi - 6\pi^2}{12(2+\pi)} \cdot \frac{qa^4}{EI} \right]$$

5.20 Determine reaction forces and moments at the fixed ends and also the vertical deflection of the point of loading. Assume $G = 0.4E$ (Fig. 5.50).

$$\left[\begin{aligned} \textit{Ans. } M &= \frac{Pa}{2}; T = 0.387\, Pa \\ \delta &= 0.711 \frac{Pa^3}{EI} \end{aligned} \right]$$

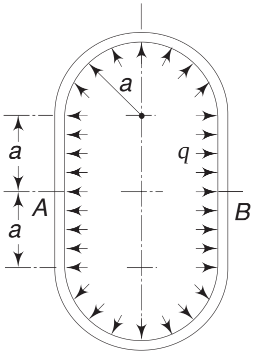

Fig. 5.49 *Problem 5.19*

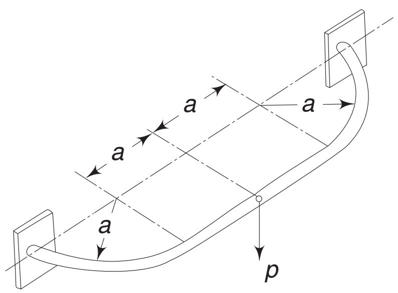

Fig. 5.50 *Problem 5.20*

5.21 A semi-circular member shown in Fig. 5.51 is subjected to a torque T at A. Determine the reactive moments at the built-in ends B and C. Also determine the vertical deflection of A.

$$\left[\begin{aligned} \textit{Ans. } M &= \frac{T}{2}; \text{Torque} = -\frac{T}{9\pi} \\ \delta_V &= \frac{R^2 T}{8EI}\left(\frac{9\pi}{4} + \frac{1}{\pi} - 5\right) \end{aligned} \right]$$

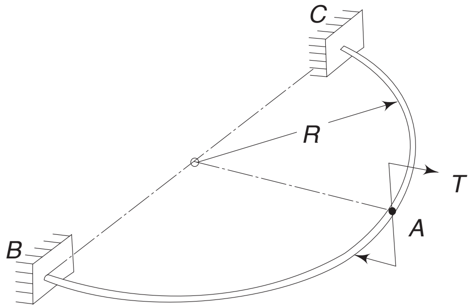

Fig. 5.51 *Problem 5.21*

5.22 In Example 5.12 determine the change in the horizontal diameter

$$\left[Ans.\ \delta_h = -\frac{Pr^3}{EI}\left(\frac{2}{\pi} - \frac{1}{2}\right) \right]$$

CHAPTER
6

Bending of Beams

6.1 INTRODUCTION

In this chapter we shall consider the stresses in and deflections of beams having a general cross-section subjected to bending. In general, the moments causing bending are due to lateral forces acting on the beams. These lateral forces, in addition to causing bending or flexural stresses in transverse sections of the beams, also induce shear stresses.

Flexural stresses are normal to the section. The effects of transverse shear stresses will be discussed in Sec. 6.4–6.6. Because of pure bending moments, only normal stresses are induced. In elementary strength of materials only beams having an axis of symmetry are usually considered. Figure 6.1 shows an initially straight beam having a vertical section of symmetry and subjected to a bending moment acting in this plane of symmetry.

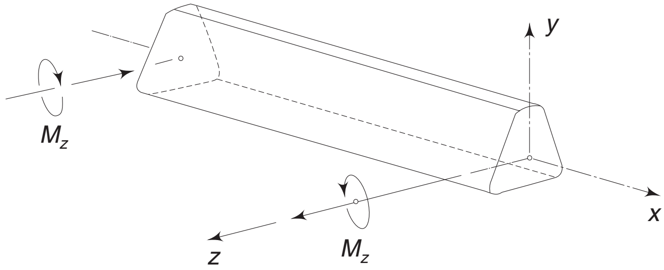

Fig. 6.1 *Beam with a vertical section of symmetry subjected to bending*

The plane of symmetry is the *xy* plane and the bending moment M_z acts in this plane. Owing to symmetry the beam bends in the *xy* plane. Assuming that the sections that are plane before bending remain so after bending, the flexural stress σ_x is obtained in elementary strength of materials as

$$\sigma_x = -\frac{M_z y}{I_z} \tag{6.1}$$

The origin of the co-ordinates coincides with the centroid of the cross-section and the *z* axis coincides with the neutral axis. The minus sign is to take care of the sign of the stress. A positive bending moment M_z, as shown, produces a compressive stress at a point with the positive y co-ordinate. I_z is the area moment of inertia about the neutral axis which passes through the centroid. Further, if E is the Young's modulus of the beam material and R the radius of curvature of the bent beam, the equations from elementary strength of materials give,

$$\frac{M_z}{I_z} = -\frac{\sigma_x}{y} = \frac{E}{R} \tag{6.2}$$

The above set of equation is usually called Euler–Bernoulli equations or Navier–Bernoulli equations.

6.2 STRAIGHT BEAMS AND ASYMMETRICAL BENDING

Now we shall consider the bending of initially straight beams having a uniform cross-section. There are three general methods of solving this problem. We shall consider each one separately. When the bending moment acts in the plane of symmetry, the beam is said to be under symmetrical bending. Otherwise it is said to be under asymmetrical bending.

Method 1 Figure 6.2 shows a beam subjected to a pure bending moment M_z lying in the xy plane. The moment is shown vectorially. The origin O is taken at the centroid of the cross-section. The x axis is along the axis of the beam and the z axis is chosen to coincide with the moment vector. It is once again assumed that sections that are plane before bending remain plane after bending. This is usually known as the Euler–Bernoulli hypothesis. This means that the cross-section will rotate about an axis such that one part of the section will be subjected to tensile stresses and the other part above this axis will be subjected to compression. Points lying on this axis will not experience any stress and consequently this axis is the neutral axis. In Fig. 6.2(b) this is represented by BB and it can be shown that it passes through the centroid O. For this, consider a small area ΔA lying at a distance y' from BB. Since the cross-section rotates about BB during bending, the stretch or contraction of any fibre will be proportional to the perpendicular distance from BB, Hence, the strain in any fibre is

$$\varepsilon_x = k'y'$$

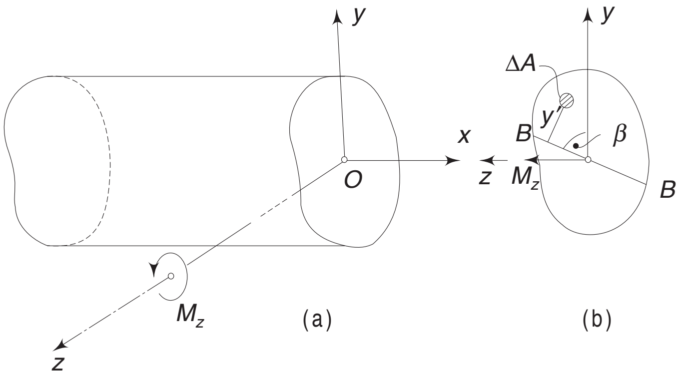

Fig. 6.2 *Beam with a general section subjected to bending*

where k' is some constant. Assuming only σ_x to be acting and $\sigma_y = \sigma_z = 0$, from Hooke's law,

$$\sigma_x = k'Ey' = ky' \tag{6.3}$$

where k is an appropriate constant. The force acting on ΔA is therefore,

$$\Delta F_x = ky' \, \Delta A$$

For equilibrium, the resultant normal force acting over the cross-section must be equal to zero. Hence, integrating the above equation over the area of the section,

$$k \iint y' \, dA = 0 \tag{6.4}$$

The above equation shows that the first moment of the area about BB is zero, which means that BB is a centroidal axis.

It is important to observe that the beam in general will not bend in the plane of the bending moment and the neutral axis BB will not be along the applied moment vector M_z. The neutral axis BB in general will be inclined at an angle β to the y axis. Next, we take moments of the normal stress distribution about the y and z axes. The moment about the y axis must vanish and the moment about the z axis should be equal to $-M_z$. The minus sign is because a positive stress at a positive (y, z) point produces a moment vector in the negative z direction. Hence

$$\iint \sigma_x z \, dA = \iint ky'z \, dA = 0 \tag{6.5a}$$

$$\iint \sigma_x y \, dA = \iint ky'y \, dA = -M_z \tag{6.5b}$$

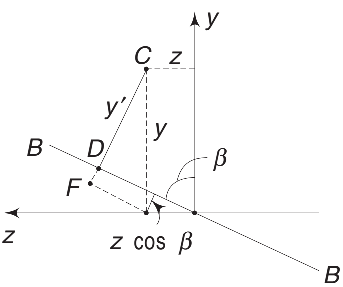

Fig. 6.3 *Location of neutral axis and distance y′ of point C from it*

y' can now be expressed in terms of y and z coordinates (Fig. 6.3) as

$$y' = CF - DF$$
$$= y \sin \beta - z \cos \beta$$

Substituting this in Eqs (6.5)

$$k \iint (yz \sin \beta - z^2 \cos \beta) \, dA = 0$$

and $$k \iint (y^2 \sin \beta - yz \cos \beta) \, dA = -M_z$$

i.e. $$I_{yz} \sin \beta - I_y \cos \beta = 0 \tag{6.6a}$$

and $$k(I_{yz} \cos \beta - I_z \sin \beta) = M_z \tag{6.6b}$$

From the first equation

$$\tan\beta = \frac{I_y}{I_{yz}} \tag{6.7}$$

This gives the location of the neutral axis BB.

Substituting for k from Eq. (6.6b) in Eq. (6.3)

$$\sigma_x = \frac{M_z \left(y \sin \beta - z \cos \beta\right)}{I_{yz} \cos \beta - I_z \sin \beta}$$

$$= \frac{y \tan \beta - z}{I_{yz} - I_z \tan \beta} M_z$$

Substituting for tan β from Eq. (6.7),

$$\sigma_x = \frac{yI_y - zI_{yz}}{I_{yz}^2 - I_y I_z} M_z \tag{6.8}$$

The above equation helps us to calculate the normal stress due to bending. In summary, we conclude that when a beam with a general cross-section is subjected to a pure bending moment M_z, the beam bends in a plane which in general does not coincide with the plane of the moment. The neutral axis is inclined at an angle β to the y axis such that tan $\beta = I_y/I_{yz}$. The stress at any point (y, z) is given by Eq. (6.8).

Method 2 we observe from Eq. (6.7) that $\beta = 90°$ when $I_{yz} = 0$, i.e. if the y and z axes happen to be the principal axes of the cross-section. This means that if the y and z axes are the principal axes and the bending moment acts in the xy plane (i.e. the moment vector M_z is along one of the principal axes), the beam bends in the plane of the moment with the neutral axis coinciding with the z axis. Equation (6.8) then reduces to

$$\sigma_x = -\frac{M_z y}{I_z}$$

This is similar to the elementary flexure formula which is valid for symmetrical bending. This is so because for a symmetrical section, the principal axes coincide with the axes of symmetry. So, an alternative method of solving the problem is to determine the principal axes of the section; next, to resolve the bending moment into components along these axes, and then to apply the elementary flexure formula. This procedure is shown in Fig. 6.4.

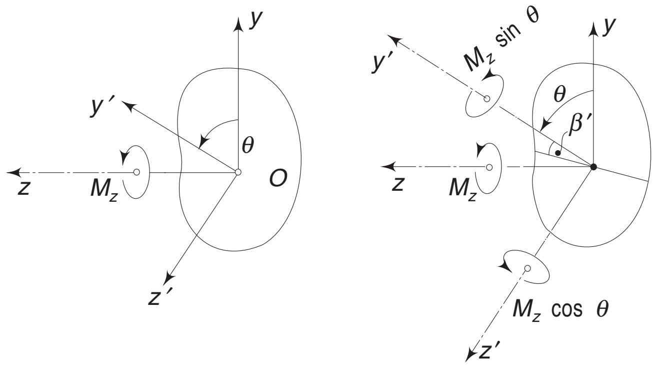

Fig. 6.4 *Resolution of bending moment vector along principal axes*

y and z axes are a set of arbitrary centroidal axes in the section. The bending moment M acts in the xy plane with the moment vector along the z axis. The principal axes Oy' and Oz' are inclined such that

$$\tan 2\theta = \frac{2I_{yz}}{I_z - I_y}$$

The moment resolved along the principal axes Oy' and Oz' are $M_y' = M_z \sin\theta$ and $M_z' = M_z \cos\theta$. For each of these moments, the elementary flexure formula can be used. With the principle of superposition,

$$\sigma_x = \frac{M_{y'}\, z'}{I_{y'}} - \frac{M_{z'}\, y'}{I_{z'}} \tag{6.9}$$

It is important to observe that with the positive axes chosen as in Fig. 6.4, a point with a positive y coordinate will be under compressive stress for positive $M_z' = M_z \cos\theta$. Hence, a minus sign is used in the equation.

The neutral axis is determined by equating σ_x to zero, i.e.

$$\frac{M_{y'}\, z'}{I_{y'}} - \frac{M_{z'}\, y'}{I_{z'}} = 0$$

or

$$\frac{z'}{y'} = \tan\beta' = \frac{M_{z'}\, I_{y'}}{M_{y'}\, I_{z'}} \tag{6.10}$$

The angle β' is with respect to the y' axis.

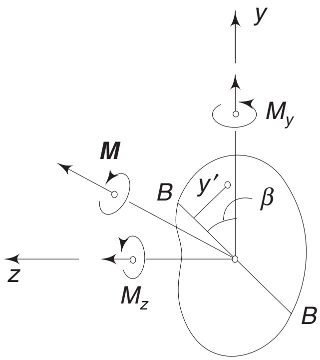

Fig. 6.5 *Resolution of bending moment vector along two arbitrary orthogonal axes*

Method 3 This is the most general method. Choose a convenient set of centroidal axes Oyz about which the moments and product of inertia can be calculated easily. Let $\boldsymbol{M}$ be the applied moment vector (Fig. 6.5).

Resolve the moment vector $\boldsymbol{M}$ into two components M_y and M_z along the y and z axes respectively. We assume the Euler-Bernoulli hypothesis, according to which the sections that were plane before bending remain plane after bending. Hence, the cross-section will rotate about an axis, such as BB. Consequently, the strain at any point in the cross-section will be proportional to the distance from the neutral axis BB.

$$\varepsilon_x = k'y'$$

Assuming that only σ_x is non-zero,

$$\sigma_x = Ek'y' = ky' \tag{a}$$

where k is some constant. For equilibirum, the total force over the cross-section should be equal to zero, since only a moment is acting.

$$\iint \sigma_x\, dA = k \iint y'\, dA = 0$$

As before, this means that the neutral axis passes through the centroid O. Let β be the angle between the neutral axis and the y axis. From geometry (Fig. 6.3).

$$y' = y \sin\beta - z\cos\beta \tag{b}$$

For equilibrium, the moments of the forces about the axes should yield

$$\iint \sigma_x z\, dA = \iint k y' z\, dA = M_y$$

$$\iint \sigma_x y\, dA = \iint k y' y\, dA = -M_z$$

Substituting for y'

$$k \iint \left(yz \sin \beta - z^2 \cos \beta \right) dA = M_y$$

$$k \iint \left(y^2 \sin \beta - yz \cos \beta \right) dA = -M_z$$

i.e. $$k \left(I_{yz} \sin \beta - I_y \cos \beta \right) = M_y \tag{6.11}$$

and $$k \left(I_z \sin \beta - I_{yz} \cos \beta \right) = -M_z \tag{6.12}$$

The above two equations can be solved for k and β. Dividing one by the other

$$\frac{I_{yz} \sin \beta - I_y \cos \beta}{I_z \sin \beta - I_{yz} \cos \beta} = -\frac{M_y}{M_z}$$

or $$\frac{I_{yz} \tan \beta - I_y}{I_z \tan \beta - I_{yz}} = -\frac{M_y}{M_z}$$

i.e. $$\tan \beta = \frac{I_y M_z + I_{yz} M_y}{I_{yz} M_z + I_z M_y} \tag{6.13}$$

This gives the location of the neutral axis BB. Next, substituting for k from Eq. (6.11) into equations (a) and (b)

$$\sigma_x = \frac{M_y \left(y \sin \beta - z \cos \beta \right)}{I_{yz} \sin \beta - I_y \cos \beta}$$

$$= \frac{M_y \left(y \tan \beta - z \right)}{I_{yz} \tan \beta - I_y}$$

Substituting for $\tan \beta$ from Eq. (6.13)

$$\sigma_x = \frac{M_z \left(y I_y - z I_{yz} \right) + M_y \left(y I_{yz} - z I_z \right)}{I_{yz}^2 - I_y I_z} \tag{6.14}$$

When $M_y = 0$ the above equation for σ_x becomes equivalent to Eq. (6.8).

In recapitulation we have the following three methods to solve unsymmetrical bending.

Method 1 Let $\boldsymbol{M}$ be the applied moment vector.

Choose a centroidal set of axes Oyz such that the z axis is along the $\boldsymbol{M}$ vector. The stress σ_x at any point (y, z) is then given by Eq. (6.8). The neutral axis is given by Eq. (6.7).

Method 2 Let $\boldsymbol{M}$ be the applied moment vector.

Choose a centroidal set of axes $Oy'z'$, such that the y' and z' axes are the principal axes. Resolve the moment into components M_y' and M_z' along the principal axes. Then the normal stress σ_x at any point (y', z') is given by Eq. (6.9) and the orientation of the neutral axis is given by Eq. (6.10).

Method 3 Choose a convenient set of centroidal axes Oyz about which the product and moments of inertia can easily be calculated. Resolve the applied moment $\boldsymbol{M}$ into components M_y and M_z. The normal stress σ_x and the orientation of the neutral axis are given by Eqs (6.14) and (6.13) respectively.

Example 6.1 *A cantilever beam of rectangular section is subjected to a load of 1000 N (102 kgf) which is inclined at an angle of 30° to the vertical. What is the stress due to bending at point D (Fig. 6.6) near the built-in-end?*

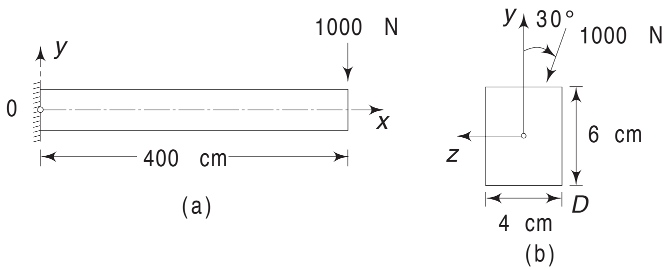

Fig. 6.6 *Example 6.1*

Solution For the section, y and z axes are symmetrical axes and hence these are also the principal axes. The force can be resolved into two components 1000 cos 30° along the vertical axis and 1000 sin 30° along the z axis. The force along the vertical axis produces a negative moment M_z (moment vector in negative z direction).

$$M_z = -(1000 \cos 30°)\, 400 = -400{,}000 \cos 30° \text{ N cm}$$

The horizontal component also produces a negative moment about the y axis, such that

$$M_y = -(1000 \sin 30°)\, 400 = -400{,}000 \sin 30° \text{ N cm}$$

The coordinates of point D are $(y, z) = (-3, -2)$. Hence, the normal stress at D from Eq. (6.9) is

$$\sigma_x = \frac{M_y z}{I_y} - \frac{M_z y}{I_z}$$

$$= \left(-400{,}000 \sin 30^\circ\right)\frac{(-2)}{I_y} - \left(-400{,}000 \cos 30^\circ\right)\frac{(-3)}{I_z}$$

$$= 400{,}000\left(\frac{2 \sin 30°}{I_y} - \frac{3 \cos 30°}{I_z}\right)$$

$$I_y = \frac{6 \times 4^3}{12} = 32 \text{ cm}^4, \qquad I_z = \frac{4 \times 6^3}{12} = 72 \text{ cm}^4$$

$$\therefore \qquad \sigma_x = 400{,}000\left(\frac{2}{2 \times 32} - \frac{3\sqrt{3}}{2 \times 72}\right)$$

$$= -1934 \text{ N/cm}^2 = -19340 \text{ kPa } (= -197 \text{ kgf/cm}^2)$$

Example 6.2 *A beam of equal-leg angle section, shown in Fig. 6.7, is subjected to its own weight. Determine the stress at point A near the built-in section. It is given that the beam weighs 1.48 N/cm (= 0.151 kgf/cm). The principal moments of inertia are 284 cm^4 and 74.1 cm^4.*

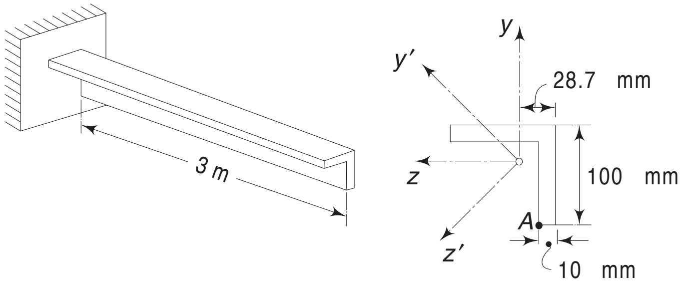

Fig. 6.7 *Example 6.2*

Solution The bending moment at the built-in end is

$$M_z = -\frac{wL^2}{2}$$

$$= \frac{1.48 \times 90{,}000}{2} = -66{,}000 \text{ N cm}$$

The centroid of the section is located at

$$\frac{(100 \times 10 \times 50) + (90 \times 10 \times 5)}{(100 \times 10) + (90 \times 10)} = 28.7 \text{ mm}$$

from the outer side of the vertical leg. The principal axes are the y' and z' axes. Since the member has equal legs, the z' axis is at 45° to the z axis. The components of M_z along y' and z' axes are, therefore,

$$M_{y'} = M_z \cos 45° = -47{,}100 \text{ N cm}$$
$$M_{z'} = M_z \cos 45° = -47{,}100 \text{ N cm}$$

$$\therefore \qquad \sigma_x = \frac{M_{y'}\, z'}{I_{y'}} - \frac{M_{z'}\, y'}{I_{z'}}$$

For point A

$$y = -(100 - 28.7) = -71.3 \text{ mm} = -7.13 \text{ cm}$$

and

$$z = -(28.7 - 10) = -18.7 \text{ mm} = -1.87 \text{ cm}$$

Hence,

$$y' = y\cos 45° + z\sin 45°$$
$$= -50.42 - 13.22 = -63.6 \text{ mm} = -6.36 \text{ cm}$$

and

$$z' = z\cos 45° - y\sin 45°$$
$$= -13.22 + 50.42 = +37.2 \text{ mm} = 3.72 \text{ cm}$$

$$\therefore \qquad \sigma_x = -\frac{47{,}100 \times 3.72}{74.1} - \frac{47{,}100 \times 6.36}{284}$$
$$= -2364 - 1055 = -3419 \text{ N/cm}^2 = -341{,}900 \text{ kPa}$$

Example 6.3 *Figure 6.8 shows a unsymmetrical one cell box beam with four-corner flange members A, B, C and D. Loads P_x and P_y are acting at a distance of 125 cm from the section ABCD. Determine the stresses in the flange members A and D. Assume that the sheet-metal connecting the flange members does not carry any flexual loads.*

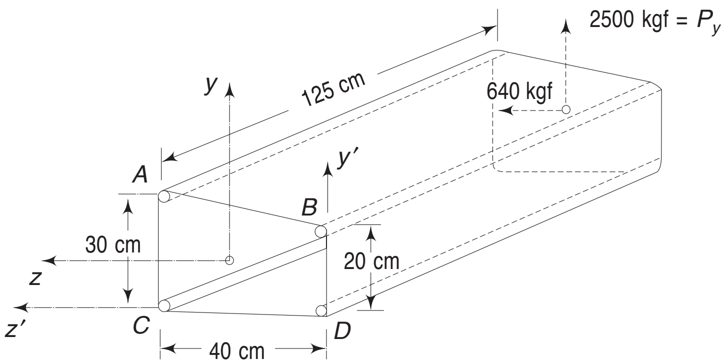

Fig. 6.8 *Example 6.3*

Solution The front face *ABCD* is assumed built-in.

Member	*Area*	*y′*	*z′*	*Ay′*	*Az′*	*y*	z
A	6.5	30	40	195	260	14.9	13.7
B	3.5	20	0	70	0	4.9	–26.3
C	5.0	0	40	0	200	–15.1	13.7
D	2.5	0	0	0	0	–15.1	–26.3
Σ =	17.5			265	460		

Therefore, the coordinates of the centroid from *D* are

$$y^* = \frac{\Sigma Ay'}{\Sigma A} = \frac{265}{17.5} = 15.1 \text{ cm}$$

$$z^* = \frac{\Sigma Az'}{\Sigma A} = \frac{460}{17.5} = 26.3 \text{ cm}$$

Member	Area	y	z	y^2	z^2	Ay^2	Az^2	Ayz
A	6.5	14.9	13.7	222	187.7	1443	1220.1	1326.8
B	3.5	4.9	–26.3	24	691.7	84	2421	–451
C	5.0	–15.1	13.7	228	187.7	1140	938.5	–1034.4
D	2.5	–15.1	–26.3	228	691.7	570	1729.3	992.8

$$\therefore \quad I_z = \Sigma Ay^2 = 3237 \text{ cm}^4$$
$$I_y = \Sigma Az^2 = 6308.9 \text{ cm}^4$$
$$I_{yz} = \Sigma Ayz = +834.2 \text{ cm}^4$$

One should be careful to observe that the loads P_y and P_z are acting at $x = -125$ cm

∴ Moment about z axis = $M_z = -312500$ kgf cm = –30646 Nm
Moment about y axis = $M_y = +80000$ kgf cm = +7845.3 Nm

From Eq. (6.14)

$$\sigma_x = \frac{-312500\left(6308.9y - 834.2z\right) + 80000\,(834.2y - 3237z)}{\left(834.2\right)^2 - \left(3237 \times 6308.9\right)}$$

$$= -96.57y - 0.09z$$

$$\therefore \quad (\sigma_x)_A = -(96.57 \times 14.9) - (0.09 \times 13.7) = -1440 \text{ kgf.cm}^2$$
$$= -141227 \text{ kPa}$$
$$(\sigma_x)_D = -(-96.57 \times 15.1) - (-0.09 \times 26.3) = +1460 \text{ kgf.cm}^2$$
$$= 143233 \text{ kPa}$$

6.3 REGARDING EULER–BERNOULLI HYPOTHESIS

We were able to solve the flexure problem because of the nature of the cross-section which remained plane after bending. It is natural to question how far this assumption is valid. In order to determine the actual deformation of an intially plane section of a beam subjected to a general loading, we will have to use the methods of the theory of elasticity. Since this is beyond the scope of this book, we shall discuss here the condition necessary for a plane section to remain plane. We have from Hooke's law

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right]$$
$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu\left(\sigma_z + \sigma_x\right)\right] \qquad \text{(c)}$$
$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right]$$

Solving the above equations for the stress σ_x we get

$$\sigma_x = \frac{\nu E}{(1+\nu)(1-2\nu)}(\varepsilon_x + \varepsilon_y + \varepsilon_z) + \frac{E}{1+\nu}\varepsilon_x$$

or from Eq. (3.15)

$$\sigma_x = \lambda J_1 + 2G\varepsilon_x \tag{6.15}$$

where λ is a constant and G is the shear modulus. According to the Euler-Bernoulli hypothesis, we have

$$\sigma_y = \sigma_z = 0$$

Hence,

$$\sigma_x = E\varepsilon_x = E\frac{\partial u_x}{\partial x} \tag{6.16a}$$

Differentiating,

$$\frac{\partial \sigma_x}{\partial x} = E\frac{\partial^2 u_x}{\partial x^2} \tag{6.16b}$$

From equilibrium equation and stress-strain relations

$$\frac{\partial \sigma_x}{\partial x} = -\frac{\partial \tau_{xy}}{\partial y} - \frac{\partial \tau_{xz}}{\partial z}$$

$$= -G\frac{\partial}{\partial y}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) - G\frac{\partial}{\partial z}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)$$

$$= -G\left(\frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right) - G\frac{\partial}{\partial x}\left(\frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right)$$

$$= -G\left(\frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right) - G\frac{\partial}{\partial x}\left(\varepsilon_y + \varepsilon_z\right) \tag{6.17a}$$

Since $\sigma_y = \sigma_z = 0$, from Eq. (c),

$$\varepsilon_y = \varepsilon_z = -\frac{\nu}{E}\sigma_x$$

Hence, Eq. (6.17a) becomes

$$\frac{\partial \sigma_x}{\partial x} = -G\left(\frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right) + \frac{2\nu G}{E}\frac{\partial \sigma_x}{\partial x}$$

i.e.

$$\frac{\partial \sigma_x}{\partial x}\left(1 - \frac{2\nu G}{E}\right) = -G\left(\frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right)$$

or

$$\frac{\partial \sigma_x}{\partial x} = -\frac{GE}{E - 2\nu G}\left(\frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right) \tag{6.17b}$$

Substituting in Eq. (6.16b),

$$E\frac{\partial^2 u_x}{\partial x^2} + \frac{GE}{E - 2\nu G}\left(\frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right) = 0$$

i.e.
$$(E-2\nu G)\frac{\partial^2 u_x}{\partial x^2}+G\left(\frac{\partial^2 u_x}{\partial y^2}+\frac{\partial^2 u_x}{\partial z^2}\right)=0$$

or
$$A\frac{\partial^2 u_x}{\partial x^2}+G\left(\frac{\partial^2 u_x}{\partial y^2}+\frac{\partial^2 u_x}{\partial z^2}\right)=0 \tag{6.18}$$

where A is a constant. From flexure formula and Eq. (6.16a)

$$\sigma_x=\frac{My}{I_z}=E\frac{\partial u_x}{\partial x} \tag{d}$$

In the above equation, M is a function of x only and y is the distance measured from the neutral axis; I_z is the moment of inertia about the neutral axis which is taken as the z axis. Then

$$E\frac{\partial^2 u_x}{\partial z^2}=\frac{y}{I_z}\frac{\partial M}{\partial x}$$

Integrating Eq. (d)

$$Eu_x=\frac{y}{I_z}\int M\,dx+\phi(y,z)$$

where ϕ is a function of y and z only. Differentiating the above expression

$$E\frac{\partial^2 u_x}{\partial y^2}=\frac{\partial^2\phi(y,z)}{\partial y^2}$$

and
$$E\frac{\partial^2 u_x}{\partial z^2}=\frac{\partial^2\phi(y,z)}{\partial z^2}$$

Substituting these in Eq. (6.18),

$$\frac{Ay}{EI_z}\frac{\partial M(x)}{\partial x}+\frac{G}{E}\left[\frac{\partial^2\phi(y,z)}{\partial y^2}+\frac{\partial^2\phi(y,z)}{\partial z^2}\right]=0$$

or
$$K_1\frac{\partial M(x)}{\partial x}=K_2\left[\frac{\partial^2\phi(y,z)}{\partial y^2}+\frac{\partial^2\phi(y,z)}{\partial z^2}\right]$$

The left-hand side quantity is a function of x alone or a constant and the right-hand side quantity is a function of y and z alone or a constant. Hence, both these quantities must be equal to a constant, i.e.

$$\frac{\partial M(x)}{\partial x}=a\ \text{constant}$$

or
$$M(x)=K_3x+K_5$$

This means that $M(x)$ can only be due to a concentrated load or a pure moment. In other words, the Euler–Bernoulli hypothesis that $\sigma_x=\dfrac{My}{I_z}$ (which is equivalent to plane sections remaining plane) will be valid only in those cases where the bending moment is a constant or varies linearly along the axis of the beam.

6.4 SHEAR CENTRE OR CENTRE OF FLEXURE

In the previous sections we considered the bending of beams subjected to pure bending moments. In practice, the beam carries loads which are transverse to the axis of the beam and which cause not only normal stresses due to flexure but also transverse shear stresses in any section. Consider the cantilever beam shown in Fig. 6.9 carrying a load at the free end. In general, this will cause both bending and twisting.

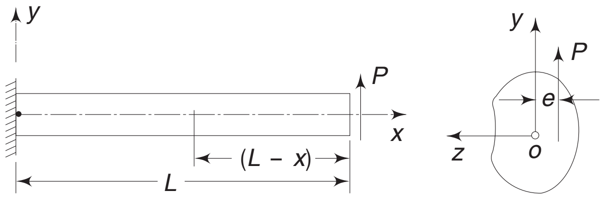

Fig. 6.9 *Cantilever beam loaded by force P*

Let *Ox* be the centroidal axis and *Oy*, *Oz* the principal axes of the section. Let the load be parallel to one of the principal axes (any general load can be resolved into components along the principal axes and each load can be treated separately). This load in general, will at any section, cause

(i) Normal stress σ_x due to flexure;

(ii) Shear stresses τ_{xy} and τ_{xz} due to the transverse nature of the loading and

(iii) Shear stresses τ_{xy} and τ_{xz} due to torsion

In obtaining a solution, we assume that

$$\sigma_x = -\frac{P(L-x)\,y}{I_z}, \qquad \sigma_y = \sigma_z = \tau_{yz} = 0 \tag{6.19}$$

This is known as St. Venant's assumption.

The values of τ_{xy} and τ_{xz} are to be determined with the equations of equilibrium and compatibility conditions. The value of σ_x as given above is derived according to the flexure formula of the previous section. The determination of τ_{xy} and τ_{xz} for a general cross-section can be quite complex. We shall not try to determine these. However, one important point should be noted. As said above, the load *P* in addition to causing bending will also twist the beam. But *P* can be applied at such a distance from the centroid that twisting does not occur. For a section with symmetry, the load has to be along the axis of symmetry to avoid twisting. For the same reason, for a beam with a general cross-section, the load *P* will have to be applied at a distance *e* from the centroid *O*. When the force *P* is parallel to the *z*-axis, a position can once again be established for which no rotation of the centroidal elements of the cross-sections occur. The point of intersection of these two lines of the bending forces is of significance. If a transverse force is applied at this point, we can resolve it into two components parallel to the *y* and *z*-axes and note from the above discussion that these components do not produce rotation of centroidal elements of the cross-sections of the beam. This point is called the shear centre of flexure or flexural centre (Fig. 6.10).

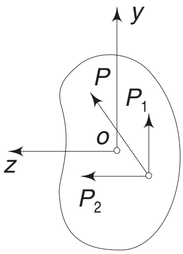

Fig. 6.10 *Load P passing through shear centre*

It is important to observe that the location of the shear centre depends only on the geometry, i.e. the shape of the section. For a section of a general shape, the location of the shear centre depends on the distribution of τ_{xy} and τ_{xz}, which, as mentioned earlier, can be quite complex. However, for thin-walled beams with open sections, approximate locations of the shear-centres can be determined by an elementary analysis, as discussed in the next section.

6.5 SHEAR STRESSES IN THIN-WALLED OPEN SECTIONS: SHEAR CENTRE

Consider a beam having a thin-walled open section subjected to a load V_y, as shown in Fig. 6.11(a). The thickness of the wall is allowed to vary. As mentioned in the previous section, the load V_y produces in general, bending, twisting and shear in the beam. Our object in this section is to locate that point through which the load V_y should act so as to cause no twist, i.e. to locate the shear centre of the section. Let us assume that load V_y is applied at the shear centre. Then there will be normal stress distribution due to bending and shear stress distribution due to vertical load. There will be no shear stress due to torsion.

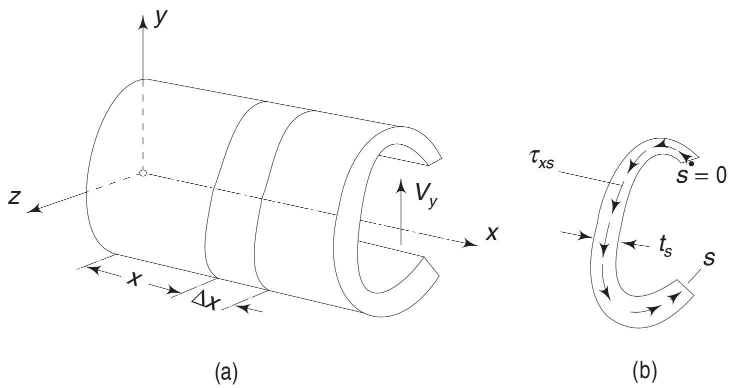

Fig. 6.11 *Thin-walled open section subjected to shear force*

The surface of the beam is not subjected to any tangential stress and hence, the boundary of the section is an unloaded boundary. Consequently, the shear stresses near the boundary cannot have a component perpendicular to the boundary. In other words, the shear stresses near the boundary lines of the section are parallel to the boundary. Since the section of the beam is thin, the shear stress can be taken to be parallel to the centre line of the section at every point as shown in Fig. 6.11(b).

Consider an element of length Δx of the beam at section x, as shown in Fig. 6.12.

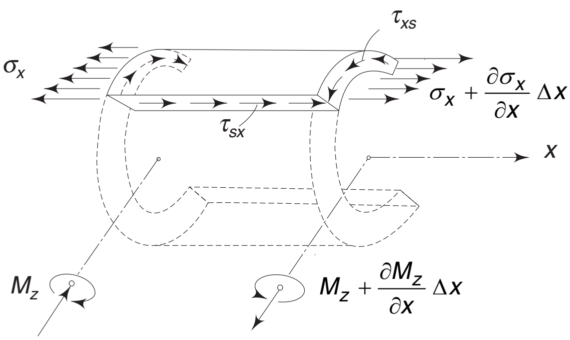

Fig. 6.12 *Free-body diagram of an elementary length of beam*

Let M_z be bending moment at section x and $M_z + \frac{\partial M_z}{\partial x}\Delta x$ the bending moment at section $x + \Delta x \cdot$ σ_x and $\sigma_x + \frac{\partial \sigma_x}{\partial x}\Delta x$, are corresponding flexural stresses at these two sections. It is important to observe that for the moments shown the normal stresses should be compressive and not as shown in the figure. However, the sign of the stress will be correctly given by Eq. (6.8). Considering a length s of the section, the unbalanced normal force is balanced by the shear stress τ_{sx} distributed along the length Δx. For equilibrium, therefore,

$$\tau_{sx}\, t_s\, \Delta x - \int_0^s \sigma_x\, t\, ds + \int_0^s \left(\sigma_x + \frac{\partial \sigma_x}{\partial x}\Delta x\right) t\, ds = 0$$

i.e.

$$\tau_{sx} = -\frac{1}{t_s}\int_0^s \frac{\partial \sigma_x}{\partial x}\, t\, ds \tag{6.20}$$

t_s is the wall thickness at s. Observing that $M_y = 0$, the normal stress σ_x is given by Eq. (6.8) as

$$\sigma_x = \frac{yI_y - zI_{yz}}{I_{yz}^2 - I_y\, I_z} M_z$$

Hence,

$$\frac{\partial \sigma_x}{\partial x} = \frac{yI_y - zI_{yz}}{I_{yz}^2 - I_y\, I_z}\frac{\partial M_z}{\partial x} \tag{6.21}$$

Recalling from elementary strength of materials $\frac{\partial M_z}{\partial x} = -V_y$, and substituting in Eq. (6.20)

$$\tau_{sx} = \frac{V_y}{t_s}\frac{1}{I_{yz}^2 - I_y\, I_z}\int_0^s (I_y y - I_{yz}\, z)\, t\, ds$$

or

$$\tau_{sx} = -\frac{V_y}{t_s\left(I_y\, I_z - I_{yz}^2\right)}\left[I_y \int_0^s yt\, ds - I_{yz}\int_0^s zt\, ds\right] \tag{6.22}$$

The first integral on the right-hand side represents the first moment of the area between $s = 0$ and s about the z axis. The second integral is the first moment of the same area between $s = 0$ and s about the y axis. Since τ_{xs} is the complementary shear stress, its value at any s is also given by Eq. (6.22).

Let Q_z be the first moment of the area between $s = 0$ and s about the z axis and Q_y the first moment of the same area about the y axis.
Then,

$$\tau_{sx} = \tau_{xs} = -\frac{V_y}{t_s\left(I_y I_z - I_{yz}^2\right)}\left[I_y Q_z - I_{yz} Q_y\right] \tag{6.23}$$

Equation (6.22) gives the shear stress distribution at section x due to the vertical load V_y acting under the explicit assumption that no twisting is caused. Hence, the shear stress distribution τ_{xs} must be statically equivalent to the load V_y. This means the following:

(i) The resultant of τ_{xs} integrated over the section area must be equal to V_y.

(ii) The moment of τ_{xs} about the centroid (or any other convenient point) must be equal to the moment of V_y about the same point. That is,

$V_y e_z = moment\ of\ \tau_{xs}$ about O

where e_z is the eccentricity or the distance of V_y from O to avoid twisting (Fig. 6.13).

If a force V_z is acting instead of V_y, we can determine the shear stress τ_{xs} at any s as

$$\tau_{xs} = -\frac{V_z}{t_s\left(I_y I_z - I_{yz}^2\right)}\left[I_z \int_0^s zt\, ds - I_{yz}\int_0^s yt\, ds\right] \tag{6.24}$$

or

$$\tau_{xs} = -\frac{V_z}{t_s\left(I_y I_z - I_{yz}^2\right)}\left[I_z Q_y - I_{yz} Q_z\right] \tag{6.25}$$

If the above shear stress distribution is due to the shear force alone and not due to twisting also, then the moment of V_z about the centroid O must be equal to the moment of τ_{xs} about the same point, i.e.

$$V_y e_z = \text{moment of } \tau_{xs} \text{ about } O$$

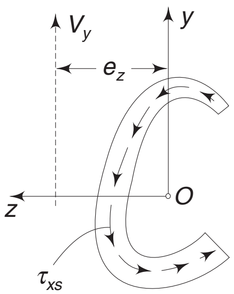

Fig. 6.13 *Location of shear centre and flow of shear stress*

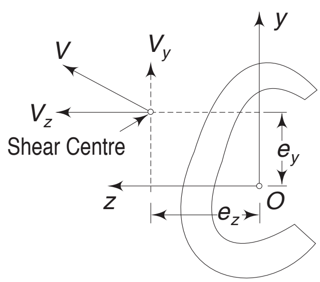

Fig. 6.14 *Location of shear centre for a general shear force*

Any arbitrary load V can be resolved into two components V_y and V_z and the resulting shear stress distribution τ_{xs} at any s is given by superposing Eqs (6.22) and (6.25). The point with coordinates (e_y, e_z), through which V_z and V_y should act to prevent the beam from twisting, is called the shear centre or the centre of flexure, as mentioned in Sec. 6.4. This is shown in Fig. 6.14.

Example 6.4 *Determine the shear stress distribution in a channel section of a cantilever beam subjected to a load F, as shown. Also, locate the shear centre of the section (Fig. 6.15).*

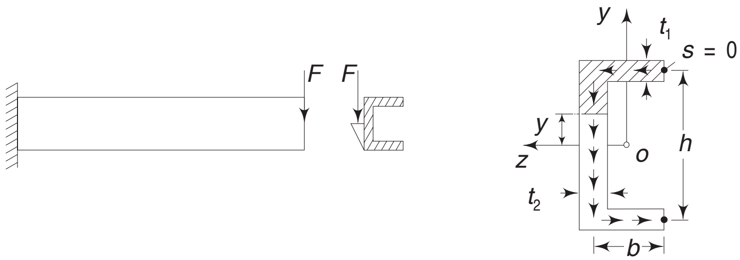

Fig. 6.15 *Example 6.4*

Solution Let Oyz be the principal axes, so that $I_{yz} = 0$. From Eq. (6.23) then, noting that F is negative,

$$\tau_{xs} = \frac{F}{t_s\, I_y\, I_z}\left(I_y\, Q_z\right)$$

or
$$\tau_{xs} = \frac{FQ_z}{t_s\, I_z}$$

where Q_z is the statical moment of the area from $s = 0$ to s about z axis. Considering the top flange, $t_s = t_1$, and the statical moment is

$$Q_z = \frac{t_1\, sh}{2}$$

Hence,
$$\tau_{xs} = \frac{Fsh}{2I_z} \quad \text{for} \quad 0 \le s < b \tag{6.26}$$

i.e. the shear stress increases linearly from $s = 0$ to $s = b$. For s in the vertical web, $t_s = t_2$, and the statical moment is the moment of the shaded area in Fig. (6.15) about the z axis, i.e.

$$Q_z = bt_1\,\frac{h}{2} + \left(\frac{h}{2} - y\right) t_2 \left[y + \frac{1}{2}\left(\frac{h}{2} - y\right)\right]$$

$$= \frac{1}{2}\left[bt_1\, h + \left(\frac{h^2}{4} - y^2\right) t_2\right]$$

Hence,
$$\tau_{xs} = \frac{F}{2t_2\, I_z}\left[bt_1\, h + \left(\frac{h^2}{4} - y^2\right) t_2\right] \quad \text{for} \; -\frac{h}{2} < y < +\frac{h}{2} \tag{6.27}$$

i.e. the shear varies parabolically from $s = b$ to $s = b + h$. For s in the horizontal flange, $t_s = t_1$ and the statical moment is

$$Q_z = bt_1\,\frac{h}{2} + 0 + (s - b - h)\, t_1 \left(-\frac{h}{2}\right)$$

$$= \left(bh + \frac{h^2}{2} - \frac{h}{2}\, s\right) t_1$$

Hence, $$\tau_{xs} = \frac{F}{2I_z}\left(bh + \frac{h^2}{2} - \frac{h}{2}s\right) \quad \text{for } 2b + h \geq s > b + h \tag{6.28}$$

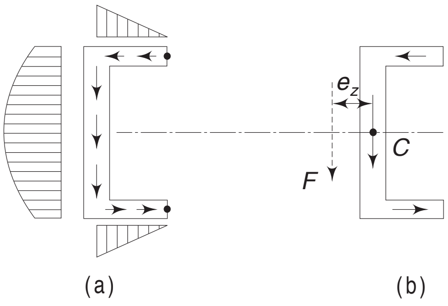

Fig. 6.16 *Example 6.4—Shear stress distribution diagrams*

i.e. the shear varies linearly. When $s = 2b + h$, i.e. the right tip of the bottom flange, the shear is zero. The variation of τ_{xs} is shown in Fig. 6.16.

This shear stress distribution should be statically equivalent to applied shear force F. It is easy to see that this is equal to F in magnitude. On integrating τ_{xs} over the area of the section, the resultant of the stress in the top and bottom flange cancel each other, and therefore, there is no horizontal resultant. Integrating τ_{xs} over the vertical web, we have

$$\int_{-h/2}^{+h/2} \tau_{xs}\, t_2\, dy = \frac{F}{2I_z}\left[\int bt_1 h\, dy + \int\left(\frac{h^2}{4} - y^2\right)t_2\, dy\right]$$

$$= \frac{F}{2I_z}\left[bt_1h^2 + \frac{h^3}{4}t_2 - \frac{h^3}{12}t_2\right]$$

$$= \frac{F}{2I_z}\left[bt_1h^2 + \frac{t_2h^3}{6}\right]$$

Now for the section

$$I_z = bt_1\frac{h^2}{4} + bt_1\frac{h^2}{4} + t_2\frac{h^3}{12}$$

$$= bt_1\frac{h^2}{2} + t_2\frac{h^3}{12} \tag{6.29}$$

Hence, $$\int_{-h/2}^{+h/2} \tau_{xs}\, t_s\, dy = F$$

Hence, the resultant of τ_{xs} over the area is equal to F. In addition, it has a moment. Taking moment about the midpoint of the vertical web [(Fig. 6.15(b)]

$$M = (\text{resultant of } \tau_{xs} \text{ in top flange}) \times \frac{h}{2}$$

$$+ (\text{resultant of } \tau_{xs} \text{ in bottom flange}) \times \frac{h}{2}$$

$$= 2\,(\text{resultant of } \tau_{xs} \text{ in top flange}) \times \frac{h}{2}$$

$$= 2\left(\text{average of } \tau_{xs} \text{ in top flange} \times \text{area} \times \frac{h}{2}\right)$$

$$= 2\left(\frac{Fbh}{4I_z} \times bt_1 \times \frac{h}{2}\right)$$

$$= \frac{Fb^2h^2t_1}{4I_z}$$

This must be equal to the moment of F about the same point. Hence, F must act at a distance e_z from C such that

$$Fe_z = \frac{Fb^2h^2t_1}{4I_z}$$

or
$$e_z = \frac{b^2h^2t_1}{4I_z}$$

Substituting for I_z from Eq. (6.29)

$$e_z = \frac{3b^2h^2t_1}{6bt_1h^2 + t_2h^3}$$

or
$$e_z = \frac{3b^2t_1}{6bt_1 + t_2h}$$

Hence, the shear centre is located at a distance e_z from C [Fig. 6.16(b)].

Example 6.5 *Determine the shear stress distribution for a circular open section under bending caused by a shear force. Locate the shear centre (Fig. 6.17).*

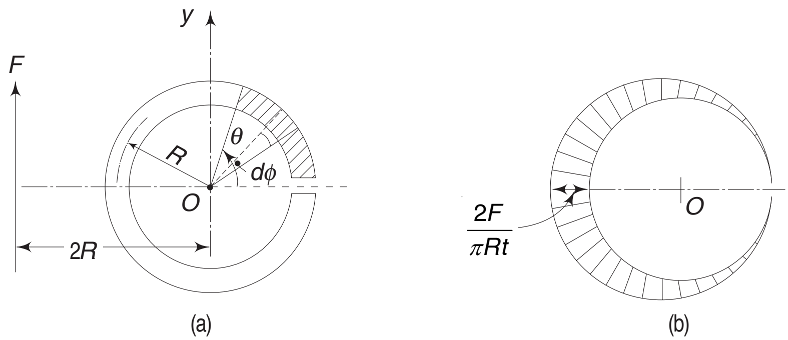

Fig. 6.17 *Example 6.5*

Solution The static moment of the crossed section is

$$Q_z = \int_0^\theta (R\,d\phi\,t)\,R\sin\phi$$

$$= R^2t\,(1 - \cos\theta)$$

Hence, from Eq. (6.23), noting that $I_{yz} = 0$, and for a vertically upward shear force F,

$$\tau_{xs} = -\frac{FQ_z}{tI_z} = -\frac{F}{tI_z} R^2 t(1 - \cos\theta)$$

But $$I_z = \pi R^3 t$$

Hence, $$\tau_{xs} = -\frac{F}{\pi Rt}(1 - \cos\theta)$$

For $\theta = 180°$ $$\tau_{xs} = -\frac{2F}{\pi Rt}$$

The distribution is shown in Fig. 6.17(b). The moment of this distribution about O is,

$$M = \int_0^{2\pi} \tau_{xs} (R\,d\theta\,t)\,R$$

$$= -\frac{F}{\pi Rt} \int_0^{2\pi} R^2 t(1 - \cos\theta)\,d\theta$$

$$= -2FR$$

This should be equal to the moment of the applied transverse force F about O. For F positive, the moment about O is negative since it is directed from $+z$ to $+y$. Hence the, force F must be applied at the shear centre C, which is at a distance of $2R$ from O.

6.6 SHEAR CENTRES FOR A FEW OTHER SECTIONS

In a thin-walled inverted T section, the distribution of shear stress due to transverse shear will be as shown in Fig. 6.18(a). The moment of this distributed stress about C is obviously zero. Hence, the shear centre for this section is C.

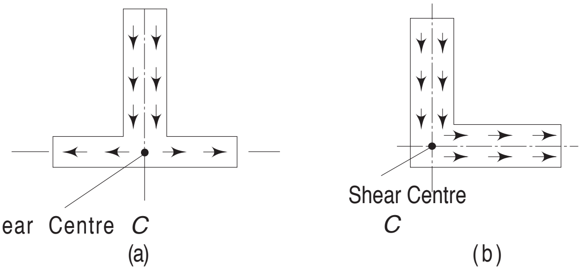

Fig. 6.18 *Location of shear centres for inverted T section and angle section*

For the angle section, the moment of the shear stresses about C is zero and hence, C is the shear centre. Figure 6.19 shows how the beams will twist if the loads are applied through the centroids of the respective sections and not through the shear centres.

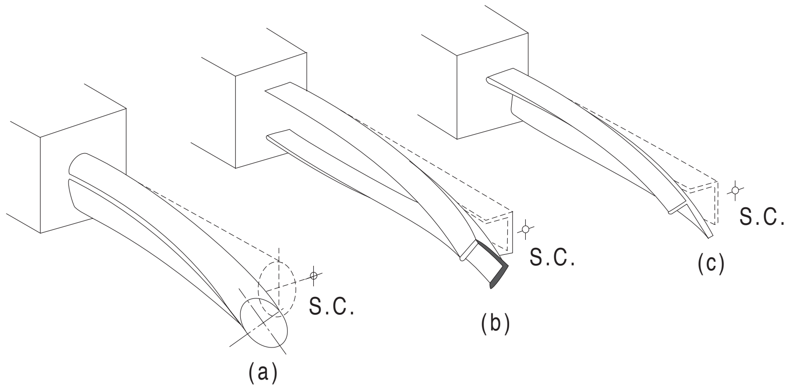

Fig. 6.19 *Twisting effect on some cross-sections if load is not applied through shear centre*

6.7 BENDING OF CURVED BEAMS (WINKLER–BACH FORMULA)

So far we have been discussing the bending of beams which are initially straight. Now we shall study the bending of beams which are initially curved. We consider the case where bending takes place in the plane of curvature. This is possible when the beam section is symmetrical about the plane of curvature and the bending moment M acts in this plane. Let ρ_0 be the initial radius of curvature of the centroidal surface. As in the case of straight beams, it is again assumed that sections which are plane before bending remain plane after bending. Hence, a transverse section rotates about an axis called the neutral axis, as shown in Fig. 6.20.

Consider an elementary length of the curved beam enclosing an angle $\Delta\phi$. Owing to the moment M, let the section AB rotate through $\delta\Delta\phi$ and occupy the position $A'B'$. The section rotates about NN, the neutral axis. SN is the trace of the neutral surface with radius of curvature r_0. Fibres above this surface get compressed and fibres below this surface get stretched. Fibres lying in the neutral surface remain unaltered. Consider a fibre at a distance y from the neutral surface. The unstretched length before bending is $(r_0 - y)\ \Delta\phi$. The change in

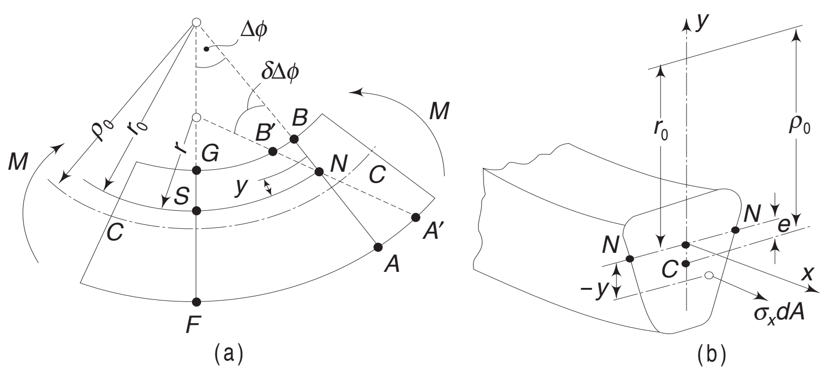

Fig. 6.20 *Geometry of bending of curved beam*

length due to bending is $y(\delta\Delta\phi)$. Noting that for the moment as shown, the strain is negative,

$$\text{strain} \equiv \varepsilon_x = -\frac{y(\delta\,\Delta\phi)}{(r_0 - y)\,\Delta\phi} \tag{6.30}$$

It is assumed here that the quantity y remains unaltered during the process of bending. The value of $(\delta\Delta\phi)/\Delta\phi$ can be obtained from Fig. 6.20(a). It is seen that

$$SN = (\Delta\phi + \delta\,\Delta\phi)\,r$$

where r is the radius of curvature of the neutral surface after bending. Also

$$SN = r_0\,\Delta\phi$$

Hence,

$$\frac{(\Delta\phi + \delta\Delta\phi)r}{\Delta\phi\, r_0} = 1$$

i.e.

$$\frac{\delta\Delta\phi}{\Delta\phi} = \frac{r_0}{r} - 1$$

$$= r_0\left(\frac{1}{r} - \frac{1}{r_0}\right) \tag{6.31}$$

Substituting in Eq. (6.30)

$$\varepsilon_x = -\frac{y}{r_0 - y}\, r_0\left(\frac{1}{r} - \frac{1}{r_0}\right) \tag{6.32a}$$

Now we shall assume that only σ_x is acting and that $\sigma_y = \sigma_z = 0$. This is similar to the Bernoulli–Euler hypothesis for the bending of straight beams. On this assumption,

$$\sigma_x = -\frac{Ey}{r_0 - y}\, r_0\left(\frac{1}{r} - \frac{1}{r_0}\right) \tag{6.32b}$$

The above expression brings out the main distinguishing feature of a curved beam. The value of y must be comparable with that of r_0, i.e. the beam must have a large curvature in which the dimensions of the cross-sections of the beam are comparable with the radius of curvature r_0. On the other hand, if the curvature (i.e. $1/r_0$) is very samll, i.e. r_0 is very large compared to y, then Eq. (6.32b) becomes

$$\sigma_x = -Ey\left(\frac{1}{r} - \frac{1}{r_0}\right)$$

With $r_0 \to \infty$, the above equation reduces to that of the straight beam. For equilibrium, the resultant of σ_x over the area should be equal to zero and the moment about NN should be equal to the applied moment M. It should be observed that the strains in fibres above the neutral axis will be numerically greater than the stains in fibres below the neutral axis. This is evident from Eq. (6.32a), since for positive y, i.e. for a fibre above the neutral axis, the denominator $(r_0 - y)$ will be less than that for a negative y. Since the resultant normal force is zero, the neutral axis gets shifted towards the centre of the curvature. For equilibrium, we have,

$$\int_A \sigma_x \, dA = -Er_0\left(\frac{1}{r}-\frac{1}{r_0}\right)\int_A \frac{y\,dA}{r_0-y} = 0$$

and

$$-\int_A \sigma_x y \, dA = +Er_0\left(\frac{1}{r}-\frac{1}{r_0}\right)\int_A \frac{y^2\,dA}{r_0-y} = M$$

From the first equation above

$$\int_A \frac{y\,dA}{r_0-y} = 0 \tag{6.33}$$

The second equation can be written as

$$+Er_0\left(\frac{1}{r}-\frac{1}{r_0}\right)\left[-\int_A y\,dA + r_0\int_A \frac{y\,dA}{r_0-y}\right] = M$$

The first integral represents the static moment of the section with respect to the neutral axis and is equal to ($-Ae$), where e is the distance of the centroid from the neutral axis NN and this moment is negative. The second integral is zero according to Eq. (6.33). Thus,

$$Er_0\left(\frac{1}{r}-\frac{1}{r_0}\right)Ae = M \tag{6.34}$$

But from Eq. (6.32)

$$Er_0\left(\frac{1}{r}-\frac{1}{r_0}\right) = -\frac{\sigma_x(r_0-y)}{y}$$

Substituting this in Eq. (6.34)

$$-\frac{\sigma_x(r_0-y)}{y}Ae = M$$

or

$$\sigma_x = -\frac{M}{Ae}\frac{y}{(r_0-y)} \tag{6.35}$$

As Eq. (6.35) shows, the normal stress varies non-linearly across the depth. The distribution is hyperbolic and one of its asymptotes coincides with the line passing through the centre of curvature, as shown in Fig. 6.21(a). The maximum stress may occur either at the top or at the bottom of the section, depending on its shapes. Equation (6.35) is often referred to as the Winkler-Bach formula.

In some texts, the origin of the coordinate system is taken at the centroid of the section instead of at the point of intersection of the neutral axis and the y axis. If the origin is taken at the centroid and y' is the distance of any fibre from this origin, then putting $y = y' - e$ and $r_0 = \rho_0 - e$, Eq. (6.35) becomes

$$\sigma_x = -\frac{M}{Ae}\frac{y'-e}{\rho_0 - e - y' + e}$$

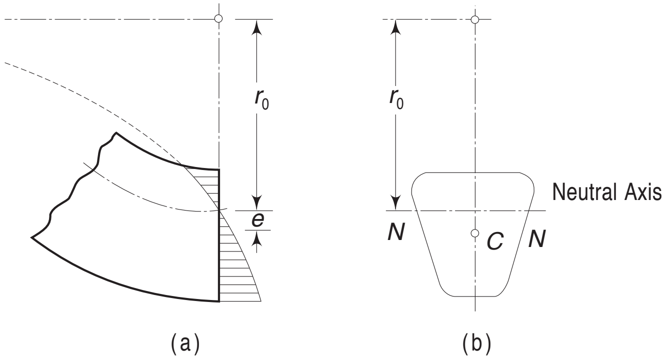

Fig. 6.21 *Distribution of normal stress and location of neutral axis*

or
$$\sigma_x = -\frac{M}{Ae}\frac{y' - e}{\rho_0 - y'} \tag{6.36}$$

To use Eq. (6.35), one requires the value of r_0. For this, consider Eq. (6.33). Introducing the new variable u

$$u = r_0 - y$$

the equation becomes

$$\int_A \frac{r_0 - u}{u} dA = 0$$

Hence,
$$r_0 = \frac{A}{\int_A dA/u} \tag{6.37}$$

The integral in the denominator represents a geometrical characteristic of the section. In other words, the values of r_0 and e are independent of the moment within elastic limit. We shall calculate these for a few of the commonly used sections.

Rectangular Section From Fig. 6.22, $dA = b\,du$ and $u = \rho_0 - y'$. Hence,

$$\int_A \frac{dA}{u} = \int_{\rho_0 - h/2}^{\rho_0 + h/2} \frac{b\,du}{u} = b \log_n \frac{\rho_0 + \frac{h}{2}}{\rho_0 - \frac{h}{2}}$$

Hence,
$$r_0 = \frac{h}{\log_n \left(\dfrac{\rho_0 + \frac{h}{2}}{\rho_0 - \frac{h}{2}}\right)} = \frac{h}{\log_n (r_2/r_1)} \tag{6.38}$$

The shift of the neutral axis from the centroid is

$$e = \rho_0 - \frac{h}{\log_n \left(\dfrac{\rho_0 + \frac{h}{2}}{\rho_0 - \frac{h}{2}}\right)} \tag{6.39a}$$

or
$$e = \rho_0 - \frac{h}{\log_n \left(\dfrac{r_2}{r_1}\right)} \tag{6.39b}$$

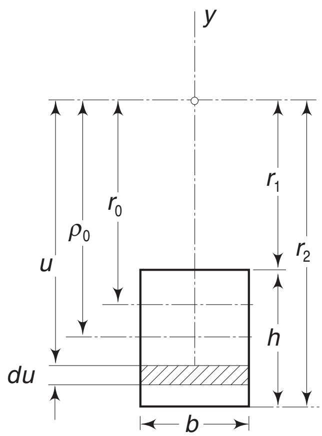

Fig. 6.22 *Parameters for a rectangular section to calculate r_0 according to Eq. (6.31)*

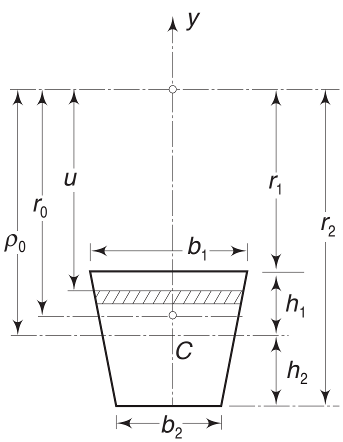

Fig. 6.23 *Parameters for a trapezoidal section to calculate r_0 according to Eq. (6.31)*

Trapezoidal Section (see Fig. 6.23) Let $h_1 + h_2 = h$. The variable width of the section is

$$b = b_2 + \frac{(b_1 - b_2)}{h}(h_2 + e + y)$$

and

$$dA = dy\left[b_2 + (b_1 - b_2)(h_2 + e + y)/h\right]$$

$$u = \rho_0 - e - y$$

$\therefore$

$$\int \frac{dA}{u} = \int_{-h_2 - e}^{h_1 - e} \left[\frac{b_2 + (b_1 - b_2)(h_2 + e + y)/h}{\rho_0 - e - y}\right] dy$$

$$= \left[b_2 + r_2(b_1 - b_2)/h\right] \log \frac{r_2}{r_1} - (b_1 - b_2)$$

When $b_1 = b_2$, the above equation reduces to that of the previous case.

$$r_0 = \frac{(b_1 + b_2)h}{2}\left\{\left[b_2 + r_2(b_1 - b_2)/h\right] \log \frac{r_2}{r_1} - (b_1 - b_2)\right\} \tag{6.40}$$

T-section (see Fig. 6.24) Proceeding as in the previous case, we obtain for the section

$$\int \frac{dA}{u} = b_1 \log \frac{r_3}{r_1} + b_2 \log \frac{r_2}{r_3} \tag{6.41}$$

I-Section For the *I*-section shown in Fig. 6.25, following the same procedure as in the preceding case,

$$\int \frac{dA}{u} = b_1 \log \frac{r_3}{r_1} + b_2 \log \frac{r_4}{r_3} + b_3 \frac{r_2}{r_4} \tag{6.42}$$

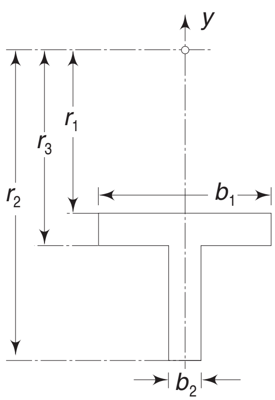

Fig. 6.24 *Parameters for T-section to calculate r_0 according to Eq. (6.31)*

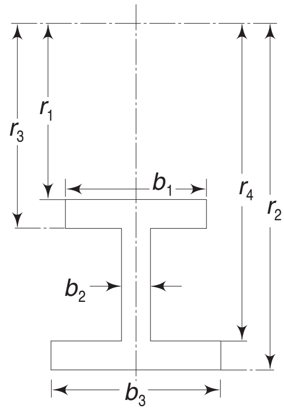

Fig. 6.25 *Parameters for I-section to calculate r_0 according to Eq. (6.31)*

Circular Section (see Fig. 6.26)

$$u = r_0 - y = (\rho_0 - e) - (a \cos \theta - e) = \rho_0 - a \cos \theta$$

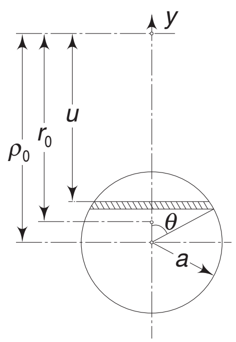

Fig. 6.26 *Parameters for a circular section to calculate r_0 according to Eq. (6.31)*

$$du = a \sin \theta \, d\theta$$

$$dA = 2a \sin \theta \, du = 2a^2 \sin^2 \theta \, d\theta$$

$$\int_A \frac{dA}{u} = \int_0^\pi 2a^2 \sin^2 \theta / (\rho_0 - a \cos \theta) \, d\theta$$

$$= 2a \int_0^\pi \frac{1 - \cos^2 \theta}{b - \cos \theta} d\theta, \quad \text{where } b = \frac{\rho_0}{a}$$

Adding and substracting $(b \cos \theta + b^2)$ to the numerator,

$$\int_A \frac{dA}{u} = 2a\pi \left[b - \left(b^2 - 1 \right)^{1/2} \right]$$

$$= 2\pi \left[\rho_0 - \left(\rho_0^2 - a^2 \right)^{1/2} \right]$$

and

$$r_0 = \frac{a^2}{2\left[\rho_0 - (\rho_0^2 - \alpha^2)^{1/2} \right]}$$

Example 6.6 *Determine the maximum tensile and maximum compressive stresses across the Sec. AA of the member loaded, as shown in Fig. 6. 27. Load P = 2000 kgf (19620 N).*

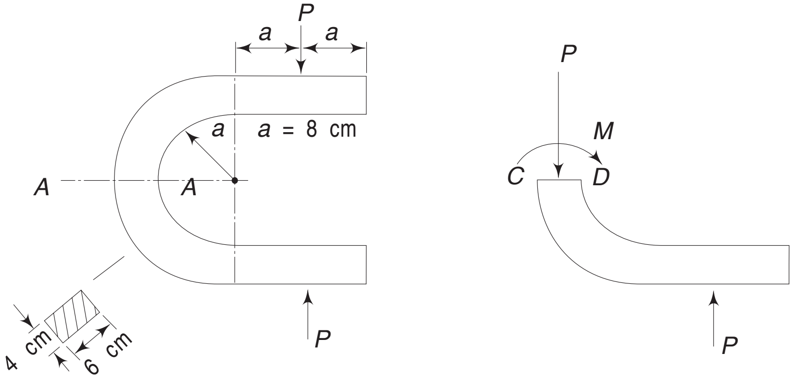

Fig. 6.27 *Example 6.6*

Solution For the section $\rho_0 = 11$ cm, $h = 6$ cm, $b = 4$ cm.

$$\therefore \quad \log \frac{\rho_0 + h/2}{\rho_0 - h/2} = \log \frac{7}{4} = 0.5596$$

From equations (6.38) and (6.39)

$$r_0 = \frac{6}{0.5596} = 10.73, \qquad e = 11 - 10.73 = 0.27$$

From Eq. (6.35), owing to bending moment M

$$\sigma'_x = -\frac{M}{Ae}\frac{y}{(r_0 - y)}$$

$$= -\frac{M}{24 \times 0.27}\frac{y}{(10.73 - y)}$$

For the problem

$$M = P\,(a + a + h/2) = 19P$$

At C, $$y = -(e + h/2) = -3.27$$

and, at D, $$y = \frac{h}{2} - e = 2.73$$

Hence, $$(\sigma'_x)_C = -\frac{19P}{24 \times 0.27} \times \frac{(-3.27)}{(10.73 + 3.27)} = 0.6848\,P$$

and $$(\sigma'_x)_D = -\frac{19P}{24 \times 0.27}\frac{2.73}{(10.73 - 2.73)} = -1.001\,P$$

The stress due to direct loading is

$$\sigma''_x = -\frac{P}{A} = -\frac{P}{24} = -0.0417\,P$$

Hence the combined stresses are

$$(\sigma_x)_C = (0.6848 - 0.0417)\,P$$
$$= 0.6431P = 129 \text{ kgf/cm}^2 \text{ (12642 kPa)}$$

and

$$(\sigma_x)_D = (-1.001 - 0.0417)\,P$$
$$= -1.0427\,P = -209 \text{ kgf/cm}^2 \text{ (20482 kPa)}$$

Example 6.7 *Determine the stress at point D of a hook (Fig. 6.28) having a trapezoidal section with the following dimensions: b_1 = 4 cm, b_2 = 1 cm, r_1 = 3 cm, r_2 = 10 cm, h = 7 cm, force P = 3000 kgf (29400 N).*

Solution For the section

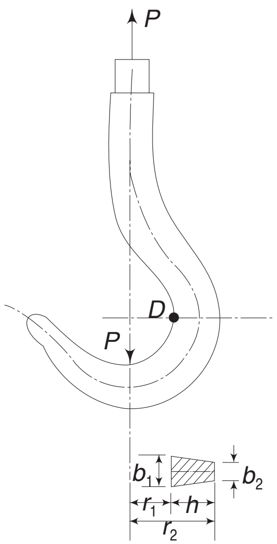

Fig. 6.28 *Example 6.7*

$$\int \frac{dA}{u} = [1 + 10(4-1)/7] \log \frac{10}{3} - (4-1)$$
$$= 3.363 \text{ cm}$$
$$A = \frac{1}{2}(b_1 + b_2)h = \frac{35}{2} = 17.5 \text{ cm}^2$$
$$\therefore \quad r_0 = A/3.363 = 17.5/3.363 = 5.204 \text{ cm}$$
$$\rho_0 = 3 + \frac{(b_1 + 2b_2)h}{3(b_1 + b_2)} = 3 + \frac{14}{5} = 5.80 \text{ cm}$$
$$\therefore \quad e = \rho_0 - r_0 = 0.596$$

The moment across section D is

$$M = -3000\,\rho_0 = -17{,}400 \text{ kgf cm (1705 Nm)}$$

The normal stress due to bending is therefore

$$(\sigma'_x)_D = -\frac{M}{Ae}\,\frac{y}{r_0 - y}$$
$$= +\frac{17{,}400}{17.5 \times 0.596} \times \frac{2.204}{5.204 - 2.2}$$
$$= 1226 \text{ kgf/cm}^2 \text{ (120,148 kPa)}$$

The normal stress due to axial loading is

$$(\sigma''_x)_D = \frac{3000}{A} = \frac{3000}{17.5} = 171 \text{ kgf/cm}^2$$

The total normal stress is therefore,

$$(\sigma_x)_D = 1397 \text{ kgf/cm}^2\text{, or } 136{,}907 \text{ kPa}$$

6.8 DEFLECTIONS OF THICK CURVED BARS

In Chapter 5, the problems of thin rings and thin curved members were analyzed using energy methods. In this section, we shall discuss a few problems involving thick rings. The energy method will be used. Consider the member shown in Fig. 6.29(a).

In the straight part of the U-ring, across any section, there is a tangential force P and a moment ($Px - M_0$). In the curved part of the member, there will

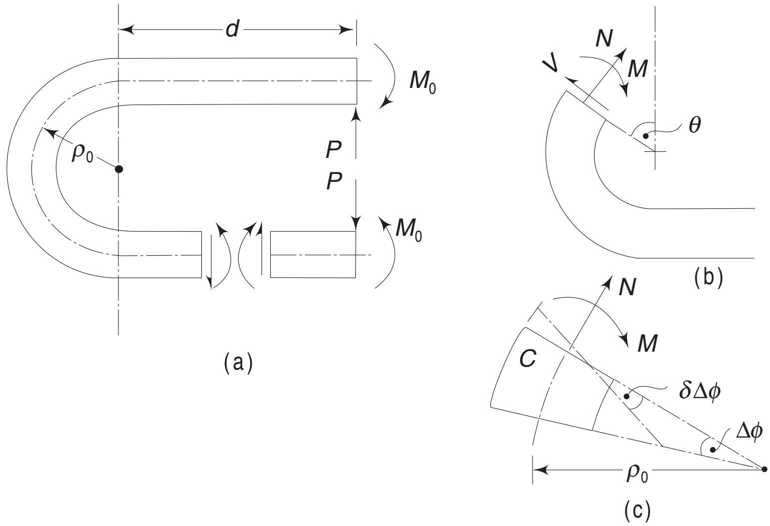

Fig. 6.29 *Geometry of deflection of a curved bar*

be a tangential force V, a normal force N and a bending moment M. Their values are

$$V = P \cos \theta$$
$$N = P \sin \theta$$
$$M = M_0 - (d + \rho_0 \sin \theta) P$$

To calculate the strain energy stored we proceed as follows (we make use of the expressions developed in Chapter 5):

(i) In the straight part of the member: Owing to the shear force V, the strain energy stored in a small length Δs is

$$\Delta U_V = \frac{\alpha V^2 \Delta s}{2AG} \tag{6.43}$$

where α is a numerical factor depending on the shape of the cross section, A is the area of the section and G is the shear modulus.

Because of the bending moment M, the energy stored is

$$\Delta U_M = \frac{M^2 \Delta s}{2EI} \tag{6.44}$$

where I is the moment of inertia about the neutral axis, which for a straight beam passes through the centroid of the section.

In general, the strain energy due to V is small as compared to that due to M.

(ii) In the curved part of the member: Owing to the shear force V, the strain energy stored in a small sectoral element, enclosing an angle $\Delta\phi$, is

$$\Delta U_V = \frac{\alpha V^2 \Delta s}{2AG} \tag{6.45}$$

If ρ_0 is the radius of curvature of the centroidal fibre, $\Delta s = \rho_0 \Delta\phi$.

Because of the normal force N, which is assumed to be acting at the centroid of the cross-section,

$$\Delta U_N = \frac{N^2 \Delta s}{2AE} \tag{6.46}$$

Owing to bending moment M, the energy stored is equal to the work done. If $\delta\,\Delta\phi$ is the change in the angle due to bending [Fig. 6.29 (c)]

$$\Delta U_M = \frac{1}{2} M\,(\delta\,\Delta\phi)$$

From Eq. (6.31),

$$\delta\Delta\phi = \Delta\phi\, r_0 \left(\frac{1}{r} - \frac{1}{r_0}\right)$$

From Eq. (6.34), substituting for the right-hand part in the above equation

$$\delta\Delta\phi = \Delta\phi \frac{M}{AeE}$$

Hence, $$\Delta U_M = \frac{M^2\,\Delta\phi}{2AeE}$$

Putting $$\Delta\phi = \frac{\Delta s}{\rho_0}$$

$$\Delta U_M = \frac{M^2\,\Delta s}{2AeE\rho_0} \tag{6.47}$$

If N is applied first and then M, owing to the rotation of the section, the centroid C [Fig. 6.29(c)] moves through a distance $\varepsilon_0\,\Delta s$, where ε_0 is the strain at C and consequently, the force N does additional work equal to

$$\Delta U_{MN} = N\,\varepsilon_0\,\Delta s$$

ε_0 from Eq. (6.35) is

$$\varepsilon_0 = \frac{\sigma_x}{E} = -\frac{M}{AeE}\frac{y_0}{(r_0 - y_0)}$$

In the above equation, M is positive, according to the convention followed (Fig. 6.20). y_0 is the distance of the centroidal fibre from the neutral axis and is equal to $-e$. Also, $\rho_0 = r_0 + e$. With these,

$$\varepsilon_0 = +\frac{M}{A\rho_0 E}$$

Hence the work done by N is

$$\Delta_{MN} = \frac{MN\,\Delta s}{A\rho_0 E} \tag{6.48}$$

The same result is obtained if M is applied first and then N. This is according to the principle of superposition, which is valid for small deformations. This can be seen by referring to Fig. 6.30.

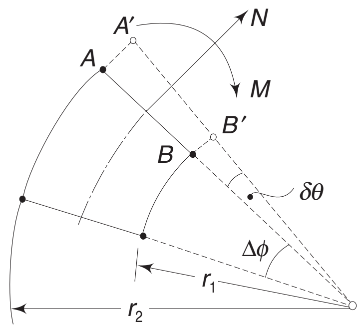

Fig. 6.30 *Deformation of a section of curved bar*

The normal force N acting across the section produces uniform strain ε_n; since the lengths of the fibres are different, face AB will not shift parallel to itself. The extension of the fibre at b will be $\varepsilon_n\, r_1\, \Delta\phi$. The angle enclosed between AB and $A'B'$ is therefore

$$\delta\theta = \frac{\varepsilon_n\, \Delta\phi\,(r_2 - r_1)}{(r_2 - r_1)} = \varepsilon_n\, \Delta\phi$$

Owing to this rotation of $A'B'$, the moment M does work equal to

$$\Delta U_{NM} = M\varepsilon_n\, \Delta\phi$$

Since

$$\varepsilon_n = \frac{N}{AE}$$

$$\Delta U_{NM} = \frac{MN}{AE}\Delta\phi$$

$$= \frac{MN\,\Delta s}{AE\rho_0}$$

For a straight beam, the work done by N when M is applied is zero since the section rotates about the neutral axis which passes through the centroid. This is also confirmed in the above expression where $\rho_0 = \infty$ for a straight beam and therefore $\Delta U_{MN} = 0$. Combining all the energies detailed above, the total strain energy is.

$$U = \int_s (\Delta U_V + \Delta U_N + \Delta U_M + \Delta U_{MN})$$

$$= \int_s \left(\frac{\alpha V^2}{2AG} + \frac{N^2}{2AE} + \frac{M^2}{2AeE\rho_0} + \frac{MN}{AE\rho_0} \right) ds \qquad (6.49)$$

For the straight part of the beam, the last expression will be zero and the third expression (which becomes indeterminate since $e = 0$ and $\rho_0 = \infty$) is replaced by $M^2/2EI$. With the strain energy calculated as above and using Castigliano's theorem, one can solve for the unknown—either the deflection or the indeterminate reaction. We shall illustrate this through an example.

Example 6.8 *A ring with a rectangular section is subjected to diametral compression, as shown in Fig. 6.31. Determine the bending moment and stress at point A of the inner radius across a section θ. r_1 and r_2 are the inner and external radii respectively.*

Solution We observe that the deformation of the ring will be symmetrical about the horizontal and vertical axes. Consequently, there will be no changes in the slopes of the vertical and horizontal faces of the ring [Fig. 6.31(b)]. We can, therefore, consider only a quadrant of the circle for the analysis. This is shown in Fig. 6.31(c). M_0 is the unknown internal moment. Its value can be determined from

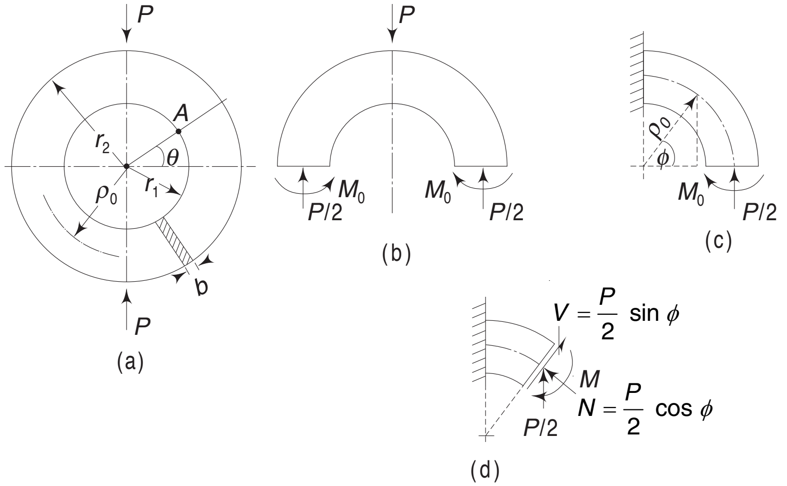

Fig. 6.31 *Example 6.8*

the condition that the change in the slope of this section is zero. We shall use Castigliano's theorem to determine this moment.

Across any section ϕ, the moment is

$$M = M_0 - \frac{P}{2}\rho_0 (1 - \cos\phi)$$

In addition, there is a normal force N and a shear force V, as shown in Fig. 6.31(d). Their values are

$$N = -\frac{P}{2}\rho_0 \cos\phi \quad \text{and} \quad V = -\frac{P}{2}\sin\phi$$

The total strain energy for the quadrant from Eq. (6.49) is

$$U = \int_0^{\pi/2} \frac{\alpha P^2 \sin^2\phi}{8AG}\rho_0\, d\phi + \int_0^{\pi/2} \frac{P^2\cos^2\phi}{8AE}\rho_0\, d\phi$$

$$+ \int_0^{\pi/2} \frac{\left[M_0 - \frac{P}{2}\rho_0 (1-\cos\phi)\right]^2}{2AeE} d\phi$$

$$- \int_0^{\pi/2} \frac{\left[M_0 - \frac{P}{2}\rho_0 (1-\cos\phi)\right] P\cos\phi}{2AE} d\phi \tag{6.50a}$$

$$= \left(\frac{\alpha P^2}{8AG} + \frac{P^2}{8AE}\right)\frac{\pi}{4}\rho_0$$

$$+ \frac{1}{2AeE}\left[M_0^2\frac{\pi}{2} + \frac{P^2}{4}\rho_0^2\left(\frac{\pi}{2} + \frac{\pi}{4} - 2\right) - M_0\rho_0 P\left(\frac{\pi}{2} - 1\right)\right]$$

$$- \frac{P}{2AE}\left(M_0 - \frac{P\rho_0}{2} + \frac{P\rho_0}{2}\frac{\pi}{4}\right) \tag{6.50b}$$

In the above expression, M_0 is still an unknown quantity. As the change in slope at the section where M is applied is zero,

$$\frac{\partial U}{\partial M_0} = \frac{1}{2AeE}\left[M_0\,\pi - \rho_0 P\left(\frac{\pi}{2}-1\right)\right] - \frac{P}{2AE} = 0$$

$$\therefore \qquad M_0 = \frac{P\rho_0}{2}\left(1 - \frac{2}{\pi} + \frac{2e}{\pi\rho_0}\right) \tag{6.51}$$

If we ignore the initial curvature of the member while calculating the strain energy, then

$$U^* = \int_0^{\pi/2} \frac{\alpha P^2 \sin^2\phi}{8AG}\,\rho_0\,d\phi + \int_0^{\pi/2} \frac{P^2\cos^2\phi}{8AE}\,\rho_0\,d\phi$$

$$+ \int_0^{\pi/2} \frac{\left[M_0 - \frac{P}{2}\rho_0\,(1-\cos\phi)\right]^2}{2EI}\,d\phi$$

and $$\frac{\partial U^*}{\partial M_0} = \frac{1}{EI}\int_0^{\pi/2}\left[M_0 - \frac{P}{2}\rho_0\,(1-\cos\phi)\right]\rho_0\,d\phi = 0$$

i.e. $$M_0\frac{\pi}{2} - \frac{P}{2}\rho_0\frac{\pi}{2} + \frac{P}{2}\rho_0 = 0$$

$$\therefore \qquad M_0 = \frac{P\rho_0}{2}\left(1 - \frac{2}{\pi}\right)$$

i.e. same as given in Eq. (6.51) with $e \to 0$ and $\rho_0 \to \infty$. Also, this moment is the same as in Example 5.12, i.e. that of a thin ring.

With the value of M_0 known, the bending moment at any section θ is obtained as

$$M = M_0 - \frac{P}{2}\rho_0\,(1-\cos\theta)$$

$$= \frac{P\rho_0}{2}\left(\cos\theta + \frac{2e}{\pi\rho_0} - \frac{2}{\pi}\right)$$

The normal stress at A can be calculated using Eq. (6.35) and adding additional stress due to the normal force N.

$$\sigma_A = -\frac{M}{Ae}\cdot\frac{y}{(r_0 - y)} + \frac{N}{A}$$

$$= -\frac{P\rho_0}{2Ae}\left(\cos\theta + \frac{2e}{\pi\rho_0} - \frac{2}{\pi}\right)\frac{y}{r_0 - y} - \frac{P\cos\theta}{2A}$$

For point A, from Eqs (6.38) and (6.39b)

$$y = \frac{h}{2} - e, \qquad r_0 = \frac{r_2 - r_1}{\log(r_2/r_1)}, \qquad e = \rho_0 - \frac{r_2 - r_1}{\log(r_2/r_1)} = \rho_0 - r_0$$

Using these

$$\sigma_A = -\frac{P}{2A}\left\{\frac{\rho_0(\pi\cos\theta - 2) + 2e}{\pi e}\frac{(h-2e)}{(2\rho_0 - h)} + \cos\theta\right\}$$

Example 6.9 *A circular ring of rectangular section, shown in Fig. 6.31, is subjected to diametral compression. Determine the change in the vertical diameter.*

Solution From Eq. (6.50b), the total energy for the complete ring is

$$U = 4\rho_0\left\{\frac{\alpha P^2\pi}{32AG} + \frac{\pi P^2}{32AE} + \frac{1}{2AeE\rho_0}\left[\frac{\pi M_0^2}{2} + \frac{\rho_0^2 P^2}{4}\left(\frac{3\pi}{4} - 2\right)\right.\right.$$

$$\left.\left. - M_0\,\rho_0\,P\left(\frac{\pi}{2} - 1\right)\right] - \frac{P}{2A\rho_0 E}\left[M_0 + \frac{P\rho_0}{2}\left(\frac{\pi}{4} - 1\right)\right]\right\}$$

where $$M_0 = \frac{\rho_0 P}{2}\left(1 - \frac{2}{\pi} + \frac{2e}{\pi\rho_0}\right)$$

$$\delta_v = \frac{\partial U}{\partial P}$$

Using the above expression for U (remembering that M is also a function of P), and simplifying

$$\delta_v = 4P\rho_0\left\{\frac{\alpha\pi}{16AG} + \frac{1}{2AE}\left(\frac{2}{\pi} - \frac{\pi}{8} - \frac{2e}{\pi\rho_0}\right) + \frac{\rho_0^2}{2AEe\rho_0}\left(\frac{\pi}{8} - \frac{1}{\pi} + \frac{e^2}{\pi\rho_0^2}\right)\right\}$$

If e is small compared to ρ_0, then

$$\delta_v \approx \frac{\alpha\pi\rho_0 P}{4AG} + \frac{2P\rho_0}{AE}\left(\frac{2}{\pi} - \frac{\pi}{8}\right) + \frac{2P\rho_0^3}{AEe\rho_0}\left(\frac{\pi}{8} - \frac{1}{\pi}\right)$$

$$= \frac{\alpha\pi P\rho_0}{4AG} + 0.488\frac{P\rho_0}{AE} + 0.15\frac{P\rho_0^2}{AEe}$$

If we assume that the ring is thin and the effect of the strain energies due to the direct force and shear force are negligible, then the chage in the vertical diameter is obtained as

$$\delta_v = \frac{P\rho_0^3}{EI}\left(\frac{\pi}{4} - \frac{2}{\pi}\right)$$

This can be seen from Eq. (6.35). When ρ_0 is large compared to y and $e \to 0$, $Ae\rho_0$ becomes equal to I according to flexure formula. Also, check with Example 5.13.

Problems

6.1 A rectangular wooden beam (Fig. 6.32) with a 10 cm × 15 cm section is used as a simply supported beam of 3 m span. It carries a uniformly distributed load of 150 kgf (1470 N) per meter. The load acts in a plane making 30° with the vertical. Calculate the maximum flexural stress at midspan and also locate the neutral axis for the same section.

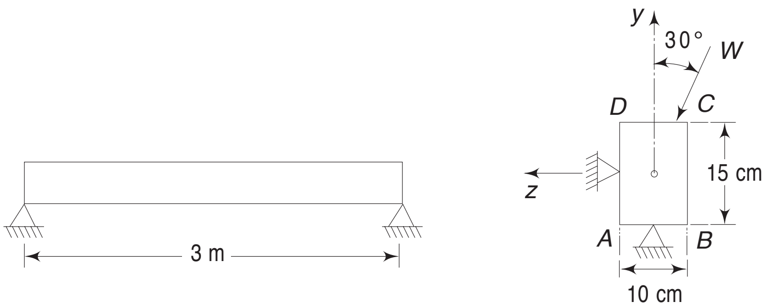

Fig. 6.32 *Problem 6.1*

[*Ans.* $\sigma_A = 73$ kgf/cm^2 = 7126 kPa
N.A cuts side *AD* such that *DN* = 1.0 cm]

6.2 A cantilever beam with a rectangular cross section, 5 cm ×10 cm which is built-in in a tilted position, carries an end load of 45 kgf (441 N), as shown in Fig. 6.33. Calculate the maximum flexural stress at the built-in end and also locate the neutral axis. The length of the cantilever is 1.2 m.

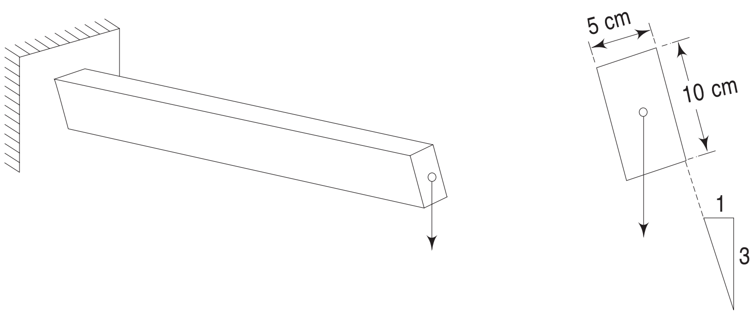

Fig. 6.33 *Problem 6.2*

[*Ans.* $\sigma = \pm 102.5$ kgf/cm^2 = 10052 kPa
N.A. is at 36.8° to the longerside]

6.3 A bar of angle section is bent by a couple *M* acting in the plane of the larger side (Fig. 6.34). Find the centroidal principal axes $Oy'z'$ and the principal moments of inertia. If M = 1.1550 kgf cm (1133 Nm), find the absolute maximum flexural stress in the section.

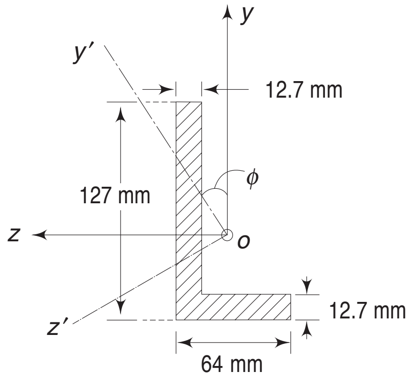

Fig. 6.34 *Problem 6.3*

[*Ans.* $\phi = \pm 14° \, 32'$
$I_{y'} = 41.9 \text{ cm}^4$; $I_{z'} = 391 \text{ cm}^4$
$\sigma_{max} = 33600$ kPa]

6.4 Determine the maximum absolute value of the normal stress due to bending and the position of the neutral axis in the dangerous section of the beam shown in Fig.6.35. Given a = 0.5 m and P = 200 kgf (1960 N). Section properties: equal legs 80 mm; centroid at 2.27 cm from the base; principal moments of inertia 116 cm^4, 30.3 cm^4; $I_z = 73.2$ cm^4.

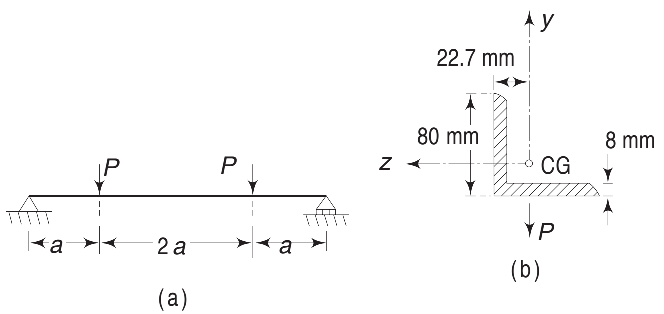

Fig. 6.35 *Problem 6.4*

[*Ans.* $\sigma = 914$ kgf/cm^2 (89640 kPa)
$\phi = 60°$ w.r.t. y axis]

6.5 Determine the maximum absolute value of the normal stress due to bending and the position of the neutral axis in the dangerous section of the beam. (Fig 6.36).

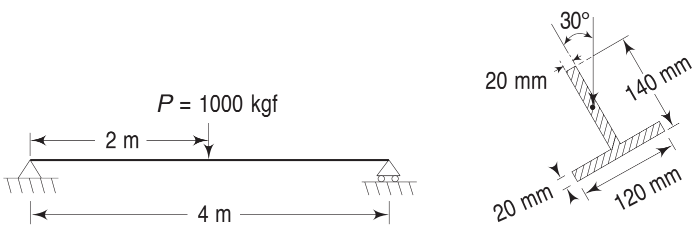

Fig. 6.36 *Problem 6.5*

[*Ans.* 1454 kgf/cm^2 (142588 kPa)
$\phi = 60.1°$ with vertical]

6.6 For the cantilever shown in Fig. 6.37, determine the maximum absolute value of the flexural stress and also locate the neutral axis at the section where this maximum stress occurs. $P = 200$ kgf (1960 N).

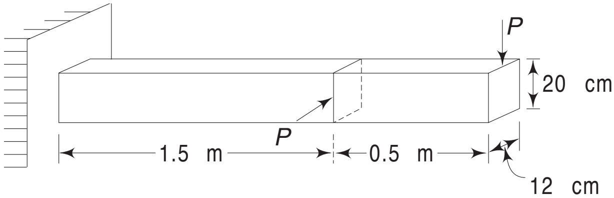

Fig. 6.37 *Problem 6.6*

[*Ans.* 112.5 kgf/cm^2 (11032 kPa)
$\phi = -25°36'$ with vertical]

6.7 A cantilever beam (Fig. 6.38) of length L has right triangular section and is loaded by P at the end. Solve for the stress at A near the built-in end. $P = 500$ kgf (4900 N), $h = 15$ cm, $b = 10$ cm and $L = 150$ cm.

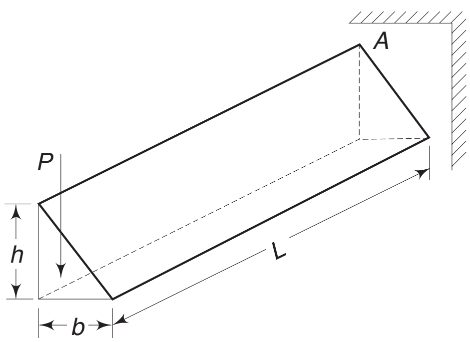

Fig. 6.38 *Problem 6.7*

[*Ans.* 2133 kgf/cm^2 (209175 kPa)]

6.8 Figure 6.39 shows an unsymmetrical beam section composed of four stringers A, B, C and D, each of equal area connected by a thin web. It is assumed that the web will not carry any bending stress. The beam section is subjected to the bending moments M_y and M_z, as indicated. Calculate the stresses in members A and D. The area of each stringer is 0.6 cm^2.

[*Ans.* $(\sigma_x)_A = -464$ kgf/cm^2 (–45503 kPa)
$(\sigma_x)_D = 448$ kgf/cm^2 (43934 kPa)]

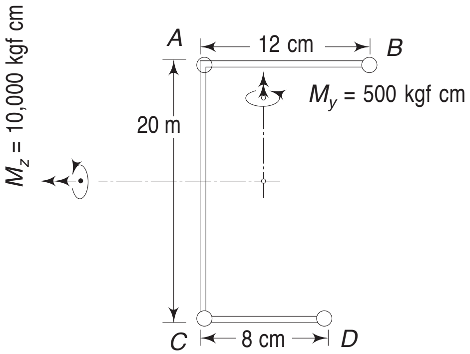

Fig. 6.39 *Problem 6.8*

6.9 In the above problem, if stringers C and D are made of magnesium alloy and stringers A and B of stainless steel, what will be the bending stresses in stringers A and D?

$$E_{\text{st st}} = 2 \times 10^6 \text{ kgf/cm}^2 \ (196 \times 10^6 \text{ kPa})$$
$$E_{\text{mg alloy}} = 0.4 \times 10^6 \text{ kgf/cm}^2 \ (39.2 \times 10^6 \text{ kPa})$$

Hint: Assume once again that sections that are plane before bending remain plane after bending. Hence, to produce the same strain, the stress will be proportional to E. Convert all the stringer areas into equivalent areas of one material. For example, the areas of stringers C and D in equivalent steel will be

$$A_C' = A_C \times \frac{E_{\text{mag}}}{E_{\text{st}}}, \qquad \text{and} \qquad A_D' = A_D \times \frac{E_{\text{mag}}}{E_{\text{st}}}$$

The areas of A and B remain unaltered. Solve the problem in the usual manner, using all equivalent steel stringers. Determine the stresses $(\sigma_x)'_A$ and $(\sigma_x)'_D$. Calculate the forces $F_A = (\sigma_x)'_A \, A'_A = (\sigma_x)'_A \, A_A$ and $F_D = (\sigma_x)'_D \, A'_D$. Now, using the original areas calculate the stress as

$$(\sigma_x)_A = (\sigma_x)'_A \, A'_A / A_A = (\sigma_x)'_A$$
$$(\sigma_x)_D = (\sigma_x)'_D \, A'_D / A_D$$

$$\left[\begin{array}{l} \textit{Ans.} \ (\sigma_x)_A = -480 \text{ kgf/cm}^2 \ (-47072 \text{ kPa}) \\ \quad\quad (\sigma_x)_D = 425.6 \text{ kgf/cm}^2 \ (41737 \text{ kPa}) \end{array} \right]$$

6.10 Show that the shear centre for the section shown in Fig. 6.40 is at $e = 4R/\pi$ measured from point 0.

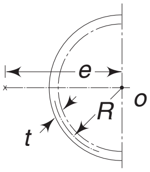

Fig. 6.40 *Problem 6.10*

6.11 For the section shown in Fig. 6.41 show that the shear centre is at a distance

$$e = R\,\frac{4(\sin\alpha - \alpha\cos\alpha)}{2\alpha - \sin 2\alpha}$$

from the centre of curvature O of the section.

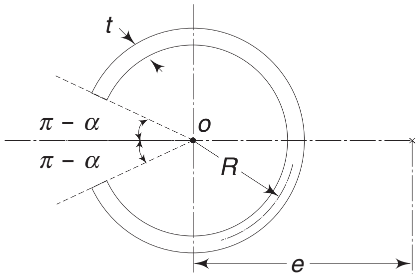

Fig. 6.41 *Problem 6.11*

6.12 Locate the shear centres from *C.G*s for the sections shown in Fig. 6.42(a), (b), and (c). In Fig. 6.42(b) the included angle is $\pi/2$.

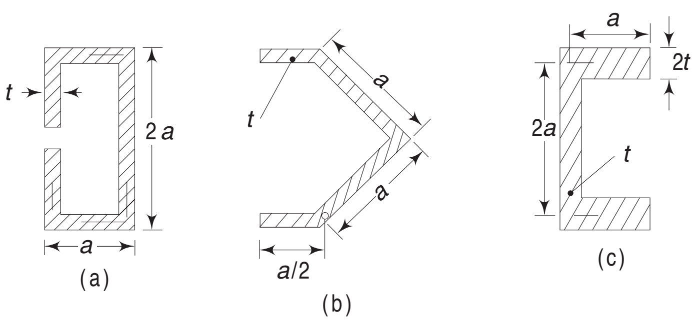

Fig. 6.42 *Problem 6.12*

[*Ans.* (a) 1.2 *a*, (b) 0.705 *a* (c) 0.76 *a*]

6.13 For the section given in Fig. 6.43, show that the shear centre is located at a distance *e* from *O* such that

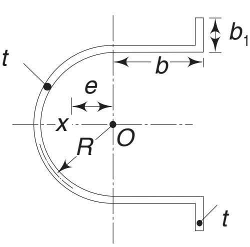

Fig. 6.43 *Problem 6.13*

$$e = \frac{A}{B}$$

where

$$A = 12 + 6\pi \frac{b + b_1}{R} + 6\left(\frac{b}{R}\right)^2 + 12\frac{b}{R}\frac{b_1}{R} + 3\pi\left(\frac{b_1}{R}\right)^2 - 4\left(\frac{b_1}{R}\right)^3 \frac{b}{R}$$

and $$B = 3\pi + 12\frac{b + b_1}{R} + 3\left(\frac{b_1}{R}\right)^2\left(4 + \frac{b_1}{R}\right)$$

Note: one can particularise this to the more familiar sections by putting b or b_1 or both equal to zero.

6.14 The open link shown in Fig. 6.44 Is loaded by forces P, each of which is equal to 1500 kgf (14,700 N). Find the maximum tensile and compressive stresses in the curved end at section AB.

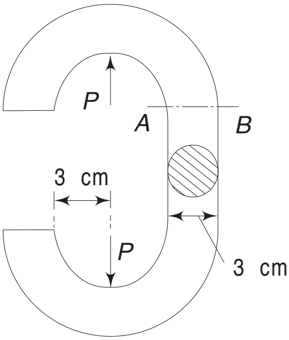

Fig. 6.44 *Problem 6.14*

$$\left[\begin{array}{ll} \textit{Ans.}\ (\sigma_x)_A & = 3591 \text{kgf/cm}^2\ (352310\text{ kPa}) \\ \quad (\sigma_x)_B & = -1796\text{ kgf/cm}^2\ (-176147\text{ kPa}) \end{array}\right]$$

6.15 A curved beam has an isosceles triangular section with the base of the triangle in the concave face. Develop the expression for r_0 in terms of the altitude h of the triangle and R the radius of curvature of the centroidal axis.

$$\left[\textit{Ans.}\ r_0 = \frac{3h^2}{2\left[(3R + 2h)\log\frac{3R + 2h}{3R - h} - 3h\right]}\right]$$

6.16 Find the maximum tensile stress in the curved part of the hook shown in Fig. 6.45. The web thickness is 1 cm.

[*Ans.* 3299 kgf/cm^2 (328680 kPa)]

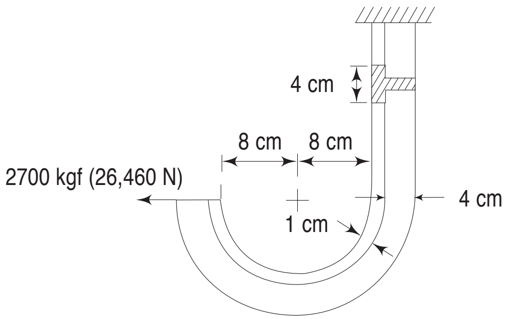

Fig. 6.45 *Problem 6.16*

6.17 Find the maximum tensile stress in the curved part of the hook shown in Fig. 6.46.

[*Ans.* $\sigma_x = 2277$ kgf/cm^2 (223300 kPa)]

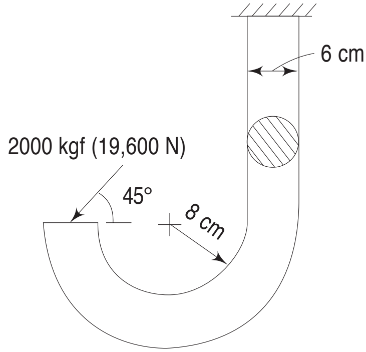

Fig. 6.46 *Problem 6.17*

6.18 Determine the ratio of the numerical value of σ_{max} and σ_{min} for a curved bar of rectangular cross-section in pure bending if $\rho_0 = 5$ cm and $h = r_2 - r_1 = 4$ cm. [*Ans.* 1.76]

6.19 Solve the previous problem if the bar is made of circular cross-section. [*Ans.* 1.89]

6.20 Determine the dimensions b_1 and b_3 of an I-section shown in Fig. 6.25, to make σ_{max} and σ_{min} numerically equal in pure bending. The other dimensions are $r_1 = 3$ cm; $r_3 = 4$ cm; $r_4 = 6$ cm; $r_2 = 7$ cm; $b_2 = 1$ cm; and $b_1 + b_3 = 5$ cm.

[*Ans.* $b_1 = 3.67$ cm, $b_3 = 1.33$ cm]

6.21 For the ring shown in Fig. 6.31 determine the changes in the horizontal diameter.

Hint: Apply two horizontal fictitious forces Q along the diameter. Calculate the total strain energy, Apply Castigliano's theorem.

$$\left[Ans.\ \delta_H = \frac{P\rho_0}{A}\left\{ -\frac{\alpha}{2G} + \frac{1}{E}\left(\frac{4}{\pi} - \frac{1}{2}\right) - \frac{1}{Ee\rho_0}\left[2e^2 - \rho_0^2\left(\frac{2}{\pi} - \frac{1}{2}\right)\right]\right\}\right]$$

CHAPTER 7 Torsion

7.1 INTRODUCTION

The torsion of circular shafts has been discussed in elementary strength of materials. There, we were able to obtain a solution to this problem under the assumption that the cross-sections of the bar under torsion remain plane and rotate without any distortion during twist. To observe this, consider the sheet shown in Fig. 7.1(a), subject to shear stress τ. The sheet deforms through an angle γ, as shown in Fig. 7.1(b).

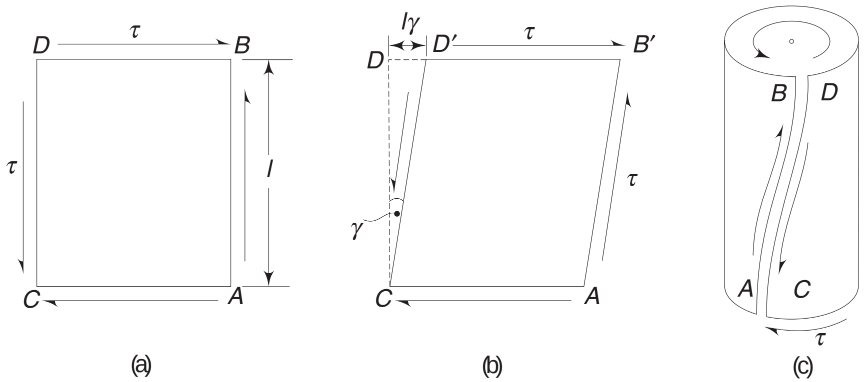

Fig. 7.1 *Deformation of a thin sheet under shear stress and the resulting tube*

If the deformed sheet is now folded to form a tube, the sides *AB* and *CD* can be joined without any discontinuity and this joined face will assume the form of a flat helix, as shown in Fig. 7.1(c). If γ is the shear strain, then from Hooke's law

$$\gamma = \frac{\tau}{G} \tag{7.1}$$

where *G* is the shear modulus. Owing to this strain, point *D* moves to *D′* [Fig. 7.1(b)], such that $DD' = l\gamma$. When the sheet is folded into a tube, the top face *BD* in Fig. 7.1(c), rotates with respect to the bottom face through an angle

$$\theta^* = \frac{l\gamma}{r} \tag{7.2}$$

where r is the radius of the tube. Substituting for γ from Eq. (7.1)

$$\theta^* = \frac{\tau}{G} \cdot \frac{l}{r}$$

or $$\frac{\theta^*}{l} = \frac{\tau}{Gr} \tag{7.3}$$

Also, the moment about the centre of the tube is

$$T = r \times 2\pi r t \tau$$

or $$T = \frac{2\pi r^3 t\tau}{r} = \frac{\tau I_p}{r}$$

i.e . $$\frac{T}{I_p} = \frac{\tau}{r} \tag{7.4}$$

where l_p is the second polar moment of area.

Equations (7.3) and (7.4), therefore, give

$$\frac{T}{I_p} = \frac{\tau}{r} = \frac{G\theta^*}{l} \tag{7.5}$$

the familiar equations from elementary strength of materials. Now one can stack a series of tubes, one inside the other and for each tube, Eq. (7.5) would be valid. These stacked tubes can form the section of a solid (or a hollow) shaft if the top face of each tube has the same rotation $G\theta^*$, i.e. if $\frac{G\theta^*}{l}$ is the same for each tube. Therefore, the ratio $\frac{\tau}{r}$ is the same for each tube, or in other words, τ varies linearly with r. Further, if T_1 is the torque on the first tube with polar moment of inertia I_{p1}, T_2 the torque on the second tube with polar moment of inertia I_{p2}, etc., then

$$\frac{T_1}{I_{p1}} = \frac{T_2}{I_{p2}} = \ldots = \frac{T_n}{I_{pn}} = \frac{T_1 + T_2 + \ldots + T_n}{I_{p1} + I_{p2} + \ldots + I_{pn}} = \frac{T}{I_p}$$

where T is the total torque on the solid (or hollow) shaft and I_p is its polar moment of inertia.

From the above analysis we observe that for circular shafts, the cross-sections remain plane before and after, and there is no distortion of the section. But, for a non-circular section, this will no longer be valid. In the case of circular shafts, the shear stresses are perpendicular to a radial line and vary linearly with the radius. We can see that both these cannot be valid for a non-circular shaft. For, if the shear stress were always perpendicular to the radius OB [Fig. 7.2(a)], it would have a component perpendicular to the boundary. This is obviously inadmissible since the surface of the shaft is unloaded and a shear stress cannot cross an unloaded boundary. Hence, at the boundary, the shear stress must be tangential to the boundary. Further, by the same argument, the shear stress at the corner of a rectangular section must be zero, since the shear stresses on both the vertical faces are zero, i.e. both boundaries are unloaded boundaries [Fig. 7.2(b)].

In order to solve the torsion problem in general, we shall adopt St. Venant's semi-inverse method. According to this method, displacements u_x, u_y and u_z are

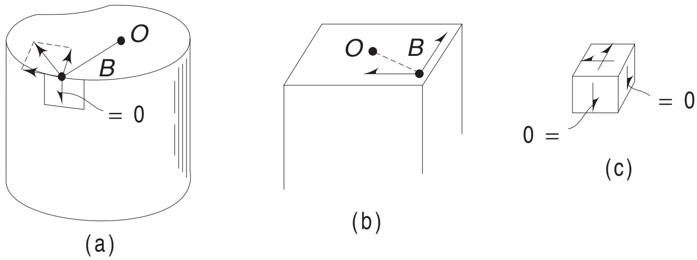

Fig. 7.2 *(a) Figure to show that shear stress must be tangential to boundary; (b) shear stress at the corner of a rectangular section being zero as shown in (c).*

assumed. The strains are then determined from strain-displacement relations [Eqs (2.18) and (2.19)]. Using Hooke's law, the stresses are then determined. Applying the equations of equilibrium and the appropriate boundary conditions, we try to identify the problem for which the assumed displacements and the associated stresses are solutions.

7.2 TORSION OF GENERAL PRISMATIC BARS–SOLID SECTIONS

We shall now consider the torsion of prismatic bars of any cross-section twisted by couples at the ends. It is assumed here that the shaft does not contain any holes parallel to the axis. In Sec. 7.12, multiply-connected sections will be discussed.

On the basis of the solution of circular shafts, we assume that the cross-sections rotate about an axis; the twist per unit length being θ. A section at distance z from the fixed end will, therefore, rotate through θz. A point $P(x, y)$ in this section will undergo a displacement $r\theta z$, as shown in Fig. 7.3. The components of this displacement are

$$u_x = -r\theta z \sin \beta$$
$$u_y = r\theta z \cos \beta$$

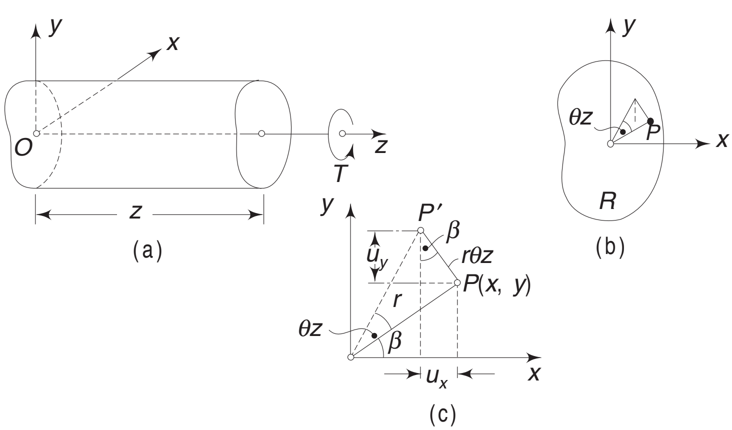

Fig. 7.3 *Prismatic bar under torsion and geometry of deformation*

From Fig. 7.3(c)

$$\sin \beta = \frac{y}{r} \quad \text{and} \quad \cos \beta = \frac{x}{r}$$

In addition to these x and y displacements, the point P may undergo a displacement u_z in z direction. This is called warping; we assume that the z displacement is a function of only (x, y) and is independent of z. This means that warping is the same for all normal cross-sections. Substituting for sin β and cos β, St. Venant's displacement components are

$$u_x = -\theta yz \tag{7.6}$$

$$\begin{aligned} u_y &= \theta xz \\ u_z &= \theta\psi(x, y) \end{aligned} \tag{7.7}$$

$\psi(x, y)$ is called the warping function. From these displacement components, we can calculate the associated strain components. We have, from Eqs (2.18) and (2.19),

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y}, \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}$$

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}, \quad \gamma_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}, \quad \gamma_{zx} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}$$

From Eqs (7.6) and (7.7)

$$\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = \gamma_{xy} = 0$$

$$\begin{aligned} \gamma_{yz} &= \theta\left(\frac{\partial \psi}{\partial y} + x\right) \\ \gamma_{zx} &= \theta\left(\frac{\partial \psi}{\partial x} - y\right) \end{aligned} \tag{7.8}$$

From Hooke's law we have

$$\sigma_x = \frac{\nu E}{(1+\nu)(1-2\nu)}\Delta + \frac{E}{1+\nu}\varepsilon_{xx}$$

$$\sigma_y = \frac{\nu E}{(1+\nu)(1-2\nu)}\Delta + \frac{E}{1+\nu}\varepsilon_{yy}$$

$$\sigma_z = \frac{\nu E}{(1+\nu)(1-2\nu)}\Delta + \frac{E}{1+\nu}\varepsilon_{zz}$$

$$\tau_{xy} = G\gamma_{xy}, \quad \tau_{yz} = G\gamma_{yz}, \quad \tau_{zx} = G\gamma_{zx}$$

where $\Delta = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$

Substituting Eq. (7.8) in the above set

$$\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0$$

$$\tau_{yz} = G\theta\left(\frac{\partial \psi}{\partial y} + x\right) \tag{7.9}$$

$$\tau_{zx} = G\theta\left(\frac{\partial \psi}{\partial x} - y\right)$$

The above stress components are the ones corresponding to the assumed displacement components. These stress components should satisfy the equations of equilibrium given by Eq. (1.65), i.e.

$$\begin{aligned}
&\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0 \\
&\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0 \\
&\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0
\end{aligned} \tag{7.10}$$

Substituting the stress components, the first two equations are satisfied identically. From the third equation, we obtain

$$G\theta\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) = 0$$

i.e.

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \nabla^2 \psi = 0 \tag{7.11}$$

Hence, the warping function ψ is harmonic (i.e. it satisfies the Laplace equation) everywhere in region R [Fig. 7.3(b)].

Now let us consider the boundary conditions. If F_x, F_y and F_z are the components of the stress on a plane with outward normal $\boldsymbol{n}$ (n_x, n_y, n_z) at a point on the surface [Fig. 7.4(a)], then from Eq. (1.9)

$$\begin{aligned}
&n_x \sigma_x + n_y\, \tau_{xy} + n_z\, \tau_{xz} = F_x \\
&n_x\, \tau_{xy} + n_y\, \sigma_y + n_z\, \tau_{yz} = F_y \\
&n_x\, \tau_{xz} + n_y\, \tau_{yz} + n_z\, \sigma_z = F_z
\end{aligned} \tag{7.12}$$

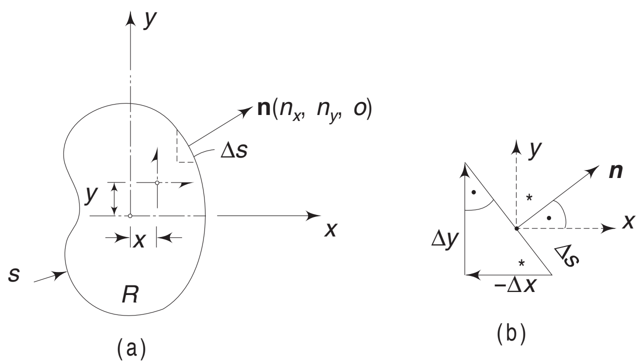

Fig. 7.4 *Cross-section of the bar and the boundary conditions*

In this case, there are no forces acting on the boundary and the normal $\boldsymbol{n}$ to the surface is perpendicular to the z-axis, i.e. $n_z \equiv 0$. Using the stress components from Eq. (7.9), we find that the first two equations in the boundary conditions are identically satisfied. The third equation yields

$$G\theta\left(\frac{\partial\psi}{\partial x}-y\right)n_x+G\theta\left(\frac{\partial\psi}{\partial y}+x\right)n_y=0$$

From Fig. 7.4(b)

$$n_x=\cos\,(n,x)=\frac{dy}{ds},\quad n_y=\cos\,(n,y)=-\frac{dx}{ds} \tag{7.13}$$

Substituting

$$\left(\frac{\partial\psi}{\partial x}-y\right)\frac{dy}{ds}-\left(\frac{\partial\psi}{\partial y}+x\right)\frac{dx}{ds}=0 \tag{7.14}$$

Therefore, each problem of torsion is reduced to the problem of finding a function ψ which is harmonic, i.e. satisfies Eq. (7.11) in region R, and satisfies Eq. (7.14) on boundary s.

Next, on the two end faces, the stresses as given by Eq. (7.9) must be equivalent to the applied torque. In addition, the resultant forces in x and y directions should vanish. The resultant force in x direction is

$$\iint_R \tau_{zx}\,dx\,dy=G\theta\iint_R\left(\frac{\partial\psi}{\partial x}-y\right)dx\,dy \tag{7.15}$$

The right-hand side integrand can be written by adding $\nabla^2\psi$ as

$$\left(\frac{\partial\psi}{\partial x}-y\right)=\left(\frac{\partial\psi}{\partial x}-y\right)+x\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)$$

since $\nabla^2\psi=0$, according to Eq. (7.11). Further,

$$\left(\frac{\partial\psi}{\partial x}-y\right)+x\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)=\frac{\partial}{\partial x}\left[x\left(\frac{\partial\psi}{\partial x}-y\right)\right]+\frac{\partial}{\partial y}\left[x\left(\frac{\partial\psi}{\partial y}+x\right)\right]$$

Hence, Eq. (7.15) becomes

$$\iint_R \tau_{zx}\,dx\,dy=G\theta\iint_R\left\{\frac{\partial}{\partial x}\left[x\left(\frac{\partial\psi}{\partial x}-y\right)\right]+\frac{\partial}{\partial y}\left[x\left(\frac{\partial\psi}{\partial y}+x\right)\right]\right\}dx\,dy$$

Using Gauss' theorem, the above surface integral can be converted into a line integral. Thus,

$$\begin{aligned}\iint_R \tau_{zx}\,dx\,dy&=G\theta\oint_S\left[x\left(\frac{\partial\psi}{\partial x}-y\right)n_x+x\left(\frac{\partial\psi}{\partial y}+x\right)n_y\right]ds\\&=G\theta\oint_S x\left[\left(\frac{\partial\psi}{\partial x}-y\right)\frac{dy}{ds}+\left(\frac{\partial\psi}{\partial y}+x\right)\frac{dx}{ds}\right]ds\\&=0\end{aligned}$$

according to the boundary condition Eq. (7.14). Similarly, we can show that

$$\iint_R \tau_{yz}\, dx\, dy = 0$$

Now coming to the moment, referring to Fig. 7.4(a) and Eq. (7.9)

$$T = \iint_R (\tau_{yz}\, x - \tau_{zx}\, y)\, dx\, dy$$

$$= G\theta \iint_R \left(x^2 + y^2 + x\frac{\partial \psi}{\partial y} - y\frac{\partial \psi}{\partial x} \right) dx\, dy$$

Writing J for the integral

$$J = \iint_R \left(x^2 + y^2 + x\frac{\partial \psi}{\partial y} - y\frac{\partial \psi}{\partial x} \right) dx\, dy \tag{7.16}$$

we have $$T = GJ\theta \tag{7.17}$$

The above equation shows that the torque T is proportional to the angle of twist per unit length with a proportionality constant GJ, which is usually called the torsional rigidity of the shaft. For a circular cross-section, the quantity J reduces to the familiar polar moment of inertia. For non-circular shafts, the product GJ is retained as the torsional rigidity.

7.3 ALTERNATIVE APPROACH

An alternative approach proposed by Prandtl leads to a simpler boundary condition as compared to Eq. (7.14). In this method, the principal unknowns are the stress components rather than the displacement components as in the previous approach. Based on the result of the torsion of the circular-shaft, let the non-vanishing stress components be τ_{zx} and τ_{yz}. The remaining stress components σ_x, σ_y, σ_z and τ_{xy} are assumed to be zero. In order to satisfy the equations of equilibrium we should have

$$\frac{\partial \tau_{zx}}{\partial z} = 0, \quad \frac{\partial \tau_{yz}}{\partial z} = 0, \quad \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0 \tag{7.18}$$

If it is assumed that in the case of pure torsion, the stresses are the same in every normal cross-section, i.e. independent of z, then the first two conditions above are automatically satisfied. In order to satisfy the third condition, we assume a function $\phi\,(x, y)$ called the stress function, such that

$$\tau_{zx} = \frac{\partial \phi}{\partial y}, \qquad \tau_{yz} = -\frac{\partial \phi}{\partial x} \tag{7.19}$$

With this stress function (called Prandtl's torsion stress function), the third condition is also satisfied. The assumed stress components, if they are to be proper elasticity solutions, have to satisfy the compatibility conditions. We can substitute these directly into the stress equations of compatibility. Alternatively, we can determine the strains corresponding to the assumed stresses and then apply the strain compatibility conditions given by Eq. (2.56). The strain components from Hooke's law are

$$\varepsilon_{xx} = 0, \qquad \varepsilon_{yy} = 0, \qquad \varepsilon_{zz} = 0 \tag{7.20}$$

$$\gamma_{xy} = 0, \qquad \gamma_{yz} = \frac{1}{G}\ \tau_{yz}, \qquad \gamma_{zx} = \frac{1}{G}\ \tau_{zx}$$

Substituting from Eq. (7.19)

$$\gamma_{yz} = -\frac{1}{G}\frac{\partial \phi}{\partial x}, \quad \text{and} \quad \gamma_{zx} = \frac{1}{G}\frac{\partial \phi}{\partial y}$$

From Eq. (2.56), the non-vanishing strain compatibility conditions are (observe that ϕ is independent of z)

$$\frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y}\right) = 0$$

$$\frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y}\right) = 0$$

i.e.

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) = 0; \quad \frac{\partial}{\partial y}\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) = 0$$

Hence,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \nabla^2 \phi = a \text{ constant } F \tag{7.21}$$

The stress function, therefore, should satisfy Poisson's equation. The constant F is yet unknown. Next, we consider the boundary conditions [Eq. (7.12)]. The first two of these are identically satisfied. The third equation gives

$$n_x \frac{\partial \phi}{\partial y} - n_y \frac{\partial \phi}{\partial x} = 0$$

Substituting for n_x and n_y from Eq. (7.13)

$$\frac{\partial \phi}{\partial y}\frac{dy}{ds} + \frac{\partial \phi}{\partial x}\frac{dx}{ds} = 0$$

i.e.

$$\frac{d\phi}{ds} = 0 \tag{7.22}$$

Therefore, ϕ is constant around the boundary. Since the stress components depend only on the differentials of ϕ, for a simply connected region, no loss of generality is involved in assuming

$$\phi = 0 \text{ on } s \tag{7.23}$$

For a multi-connected region R (i.e. a shaft having holes), certain additional conditions of compatibility are imposed. This will be discussed in Sec. 7.9.

On the two end faces, the resultants in x and y directions should vanish, and the moment about O should be equal to the applied torque T. The resultant in x direction is

$$\iint_R \tau_{zx}\, dx\, dy = \iint_R \frac{\partial \phi}{\partial y}\, dx\, dy$$

$$= \int dx \int \frac{\partial \phi}{\partial y}\, dy$$

$$= 0$$

since ϕ is constant around the boundary. Similarly, the resultant in y direction also vanishes. Regarding the moment, from Fig. 7.4(a)

$$T = \iint_R (x\tau_{zy} - y\tau_{zx})\, dx\, dy$$

$$= -\iint_R \left(x\frac{\partial \phi}{\partial x} + y\frac{\partial \phi}{\partial y} \right) dx\, dy$$

$$= -\iint_R x\frac{\partial \phi}{\partial x}\, dx\, dy - \iint_R y\frac{\partial \phi}{\partial y}\, dx\, dy$$

Integrating by parts and observing that $\phi = 0$ of the boundary, we find that each integral gives

$$\iint \phi\, dx\, dy$$

Thus $$T = 2\iint \phi\, dx\, dy \tag{7.24}$$

Hence, we observe that half the torque is due to τ_{zx} and the other half to τ_{yz}.

Thus, all differential equations and boundary conditions are satisfied if the stress function ϕ obeys Eqs (7.21), (7.23) and (7.24). But there remains an indeterminate constant in Eq. (7.21). To determine this, we observe from Eq. (7.19)

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial \tau_{zx}}{\partial y} - \frac{\partial \tau_{yz}}{\partial x}$$

$$= G\left(\frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{yz}}{\partial x} \right)$$

$$= G\left[\frac{\partial}{\partial y}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right) - \frac{\partial}{\partial x}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \right) \right]$$

$$= G\frac{\partial}{\partial z}\left(\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x} \right)$$

$$= G\frac{\partial}{\partial z}(-2\omega_z)$$

where ω_z is the rotation of the element at (x, y) about the z-axis [Eq. (2.25), Sec. 2.8]. $(\partial/\partial z)$ (ω_z) is the rotation per unit length. In this chapter, we have termed it as twist per unit length and denoted it by θ. Hence,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \nabla^2 \phi = -2G\theta \tag{7.25}$$

According to Eq. (7.19),

$$\tau_{zx} = \frac{\partial \phi}{\partial y}, \quad \tau_{yz} = -\frac{\partial \phi}{\partial x}$$

That is, the shear acting in the x direction is equal to the slope of the stress function ϕ (x, y) in the y direction. The shear stress acting in the y direction is equal to the negative of the slope of the stress function in the x direction. This condition may be generalised to determine the shear stress in any direction, as follows. Consider a line of constant ϕ in the cross-section of the bar. Let s be the contour line of ϕ = constant [Fig. 7.5(a)] along this contour

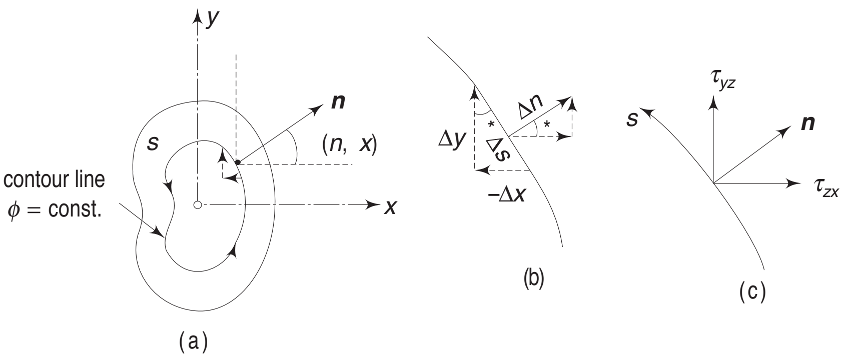

Fig. 7.5 *Cross-section of the bar and contour lines of* ϕ

$$\frac{d\phi}{ds} = 0 \tag{7.26a}$$

i.e.

$$\frac{\partial \phi}{dx}\frac{dx}{ds} + \frac{\partial \phi}{dy}\frac{dy}{ds} = 0 \tag{7.26b}$$

or

$$-\tau_{yz}\frac{dx}{ds} + \tau_{zx}\frac{dy}{ds} = 0 \tag{7.26c}$$

From Fig. 7.5(b)

$$-\frac{dx}{ds} = \cos(\boldsymbol{n}, y) = \frac{dy}{dn}$$

and

$$-\frac{dy}{ds} = \cos(\boldsymbol{n}, x) = \frac{dx}{dn}$$

where $\boldsymbol{n}$ is the outward drawn normal. Therefore, Eq. (7.26c) becomes

$$\tau_{yz}\cos(\boldsymbol{n}, y) + \tau_{zx}\cos(\boldsymbol{n}, x) = 0 \tag{7.27a}$$

From Fig. 7.5(c), the expression on the left-hand side is equal to τ_{zn}, the component of resultant shear in the direction $\boldsymbol{n}$.

Hence,

$$\tau_{zn} = 0 \tag{7.27b}$$

This means that the resultant shear at any point is along the contour line of ϕ = constant at that point. These contour lines are called lines of shearing stress. The resultant shearing stress is therefore

$$\tau_{zs} = \tau_{yz}\sin(\boldsymbol{n}, y) - \tau_{zx}\sin(\boldsymbol{n}, x)$$

$$= \tau_{yz} \cos (\boldsymbol{n}, x) - \tau_{zx} \cos (\boldsymbol{n}, y)$$

$$= \tau_{yz} \frac{dx}{dn} - \tau_{zx} \frac{dy}{dn}$$

$$= -\frac{\partial \phi}{\partial x}\frac{dx}{dn} - \frac{\partial \phi}{\partial y}\frac{dy}{dn} \tag{7.28}$$

or $$\tau_{zs} = -\frac{\partial \phi}{\partial n}$$

Thus, the magnitude of the shearing stress at a point is given by the magnitude of the slope of $\phi (x, y)$ measured normal to the tangent line, i.e. normal to the contour line at the concerned point. The above points are very important in the analysis of a torsion problem by membrane analogy, discussed in Sec. 7.7.

7.4 TORSION OF CIRCULAR AND ELLIPTICAL BARS

(i) The simplest solution to the Laplace equation (Eq. 7.11) is

$$\psi = \text{constant} = c \tag{7.29}$$

With $\psi = c$, the boundary condition given by Eq. (7.14) becomes

$$-y\frac{dy}{ds} - x\frac{dx}{ds} = 0$$

or $$\frac{d}{ds}\frac{x^2 + y^2}{2} = 0$$

i.e. $$x^2 + y^2 = \text{constant}$$

where (x, y) are the coordinates of any point on the boundary. Hence, the boundary is a circle. From Eq. (7.7), $u_z = \theta c$. From Eq. (7.16)

$$J = \iint_R (x^2 + y^2)\, dx\, dy = I_p$$

the polar moment of inertia for the section. Hence, from Eq. (7.17)

$$T = GI_p\theta$$

or $$\theta = \frac{T}{GI_p}$$

Therefore, $$u_z = \theta c = \frac{Tc}{GI_p}$$

which is a constant. Since the fixed end has zero u_z at least at one point, u_z is zero at every cross-section (other than rigid body displacement). Thus, the cross-section does not warp. The shear stresses are given by Eq. (7.9) as

$$\tau_{yz} = G\theta x = \frac{Tx}{I_p}$$

$$\tau_{zx} = -G\theta y = -\frac{Ty}{I_p}$$

Therefore, the direction of the resultant shear τ is such that, from Fig. 7.6

$$\tan\alpha = \frac{\tau_{zy}}{\tau_{zx}} = -\frac{G\theta x}{G\theta y} = -\frac{x}{y}$$

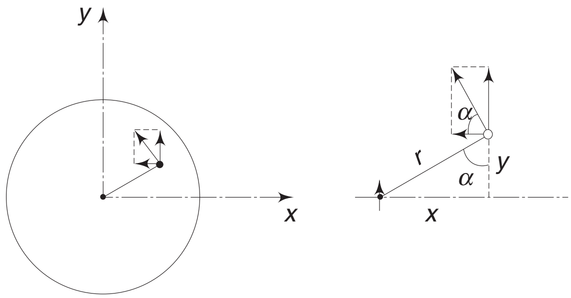

Fig. 7.6 *Torsion of a circular bar*

Hence, the resultant shear is perpendicular to the radius. Further

$$\tau^2 = \tau_{yz}^2 + \tau_{zx}^2 = \frac{T^2(x^2 + y^2)}{I_p^2}$$

or
$$\tau = \frac{Tr}{I_p}$$

where r is the radial distance of the point (x, y). Thus, all the results of the elementary analysis are justified.

(ii) The next case in the order of simplicity is to assume that

$$\psi = Axy \tag{7.30}$$

where A is a constant. This also satisfies the Laplace equation. The boundary condition, Eq. (7.14) gives,

$$(Ay - y)\frac{dy}{ds} - (Ax + x)\frac{dx}{ds} = 0$$

or
$$y(A - 1)\frac{dy}{ds} - x(A + 1)\frac{dx}{ds} = 0$$

i.e.
$$(A + 1)\,2x\frac{dx}{ds} - (A - 1)\,2y\frac{dy}{ds} = 0$$

or
$$\frac{d}{ds}\left[(A + 1)x^2 - (A - 1)y^2\right] = 0$$

which on integration, yields

$$(1 + A)\,x^2\,(1 - A)\,y^2 = \text{constant} \tag{7.31}$$

This is of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

These two are identical if

$$\frac{a^2}{b^2} = \frac{1-A}{1+A}$$

or
$$A = \frac{b^2 - a^2}{b^2 + a^2}$$

Therefore, the function

$$\psi = \frac{b^2 - a^2}{b^2 + a^2}\ xy$$

represents the warping function for an elliptic cylinder with semi-axes a and b under torsion. The value of J, as given in Eq. (7.16), is

$$J = \iint_R (x^2 + y^2 + Ax^2 - Ay^2)\ dx\ dy$$

$$= (A + 1) \iint x^2\ dx\ dy + (1 - A) \iint y^2\ dx\ dy$$

$$= (A + 1)\ I_y + (1 - A)\ I_x$$

Substituting $I_x = \dfrac{\pi ab^3}{4}$ and $I_y = \dfrac{\pi a^3 b}{4}$, one gets

$$J = \frac{\pi a^3 b^3}{a^2 + b^2}$$

Hence, from Eq. (7.17)

$$T = GJ\theta = G\theta\ \frac{\pi a^3 b^3}{a^2 + b^2}$$

or
$$\theta = \frac{T}{G}\ \frac{a^2 + b^2}{\pi a^3 b^3} \tag{7.32}$$

The shearing stresses are given by Eq. (7.9) as

$$\tau_{yz} = G\theta \left(\frac{\partial \psi}{dy} + x \right)$$

$$= T \frac{a^2 + b^2}{\pi a^3 b^3} \left(\frac{b^2 - a^2}{b^2 + a^2} + 1 \right) x$$

$$= \frac{2Tx}{\pi a^3 b} \tag{7.33a}$$

and similarly,

$$\tau_{zx} = \frac{2Ty}{\pi ab^3} \tag{7.33b}$$

The resultant shearing stress at any point (x, y) is

$$\tau = \left[\tau_{yz}^2 + \tau_{zx}^2\right]^{1/2} = \frac{2T}{\pi a^3 b^3}\left[b^4 x^2 + a^4 y^2\right]^{1/2} \tag{7.33c}$$

To determine where the maximum shear stress occurs, we substitute for x^2 from

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \text{or} \quad x^2 = a^2\left(1 - \frac{y^2}{b^2}\right)$$

giving
$$\tau = \frac{2T}{\pi a^3 b^3}\,[a^2 b^4 + a^2 (a^2 - b^2)\, y^2]^{1/2}$$

Since all terms under the radical (power 1/2) are positive, the maximum shear stress occurs when y is maximum, i.e. when $y = b$. Thus, τ_{max} occurs at the ends of the minor axis and its value is

$$\tau_{max} = \frac{2T}{\pi a^3 b^3}(a^4 b^2)^{1/2} = \frac{2T}{\pi a b^2} \tag{7.34}$$

With the warping function known, the displacement u_z can easily be determined. We have from Eq. (7.7)

$$u_z = \theta\psi = \frac{T(b^2 - a^2)}{\pi a^3 b^3 G}\, xy$$

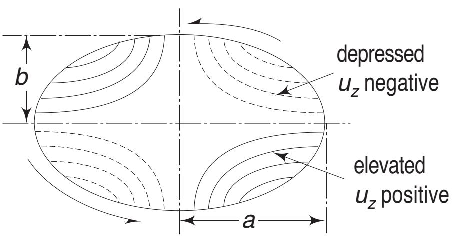

Fig. 7.7 *Cross-section of an elliptical bar and contour lines of u_z*

The contour lines giving u_z = constant are the hyperbolas shown in Fig. 7.7. For a torque T as shown, the convex portions of the cross-section, i.e. where u_z is positive, are indicated by solid lines, and the concave portions or where the surface is depressed, are shown by dotted lines. If the ends are free, there are no normal stresses. However, if one end is built-in, the warping is prevented at that end and consequently, normal stresses are induced which are positive in one quadrant and negative in another. These are similar to bending stresses and are, therefore, called the bending stresses induced because of torsion.

7.5 TORSION OF EQUILATERAL TRIANGULAR BAR

Consider the warping function

$$\psi = A(y^3 - 3x^2 y) \tag{7.35}$$

This satisfies the Laplace equation, which can easily be verified. The boundary condition given by Eq. (7.14) yields

$$(-6Axy - y)\frac{dy}{ds} - (3Ay^2 - 3Ax^2 + x)\frac{dx}{ds} = 0$$

or $$y(6Ax+1)\frac{dy}{ds}+(3Ay^2-3Ax^2+x)\frac{dx}{ds}=0$$

i.e. $$\frac{d}{ds}\left(3Axy^2-Ax^3+\frac{1}{2}x^2+\frac{1}{2}y^2\right)=0$$

Therefore,

$$A(3xy^2-x^3)+\frac{1}{2}x^2+\frac{1}{2}y^2=b \tag{7.36}$$

where b is a constant. If we put $A=-\frac{1}{6a}$ and $b=+\frac{2a^2}{3}$, Eq. (7.36) becomes

$$-\frac{1}{6a}(3xy^2-x^3)+\frac{1}{2}(x^2+y^2)-\frac{2}{3}a^2=0$$

or $$(x-\sqrt{3}\,y+2a)(x+\sqrt{3}\,y+2a)(x-a)=0 \tag{7.37}$$

Equation (7.37) is the product of the three equations of the sides of the triangle shown in Fig. 7.8. The equations of the boundary lines are

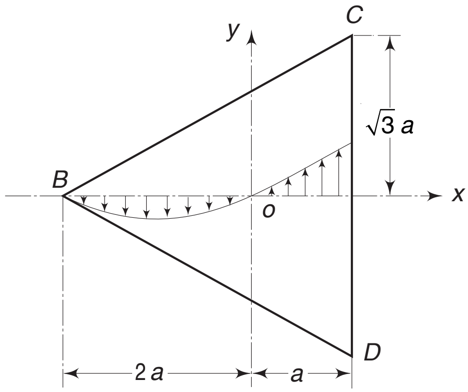

Fig. 7.8 *Cross-section of a triangular bar and plot of τ_{yz} along x-axis*

$$x-a=0 \quad \text{on } CD$$
$$x-\sqrt{3}\,y+2a=0 \quad \text{on } BC$$
$$x+\sqrt{3}\,y+2a=0 \quad \text{on } BD$$

From Eq. (7.16)

$$J=\iint_R\left[x^2+y^2+Ax\left(3y^2-3x^2\right)-Ay\left(-6xy\right)\right]dx\,dy$$

$$=\int_0^{\sqrt{3}a}dy\int_{-\sqrt{3}y-2a}^{a}\left[x^2+y^2+Ax\left(3y^2-3x^2\right)-Ay\left(-6xy\right)\right]dx$$

$$+\int_{-\sqrt{3}a}^{a}dy\int_{-\sqrt{3}y-2a}^{a}\left[x^2+y^2+Ax\left(3y^2-3x^2\right)-Ay\left(-6xy\right)\right]dx$$

$$=\frac{9\sqrt{3}}{5}a^4=\frac{3}{5}I_p \tag{7.38}$$

Therefore,

$$\theta = \frac{T}{GJ} = \frac{5}{3}\frac{T}{GI_P} \tag{7.39}$$

I_p is the polar moment of inertia about 0.

The stress components are

$$\tau_{yz} = G\theta\left(\frac{\partial\psi}{\partial y} + x\right)$$

$$= G\theta\left(3Ay^2 - 3Ax^2 + x\right)$$

$$= \frac{G\theta}{2a}\left(x^2 - y^2 + 2ax\right) \tag{7.40}$$

and

$$\tau_{zx} = G\theta\left(\frac{\partial\psi}{\partial y} - y\right)$$

$$= \frac{G\theta y}{a}\left(x - a\right) \tag{7.41}$$

The largest shear stress occurs at the middle of the sides of the triangle, with a value

$$\tau_{max} = \frac{3G\theta a}{2} \tag{7.42}$$

At the corners of the triangle, the shear stresses are zero. Along the x-axis, $\tau_{zx} = 0$ and the variation of τ_{yz} is shown in Fig. 7.8. τ_{yz} is also zero at the origin 0.

7.6 TORSION OF RECTANGULAR BARS

The torsion problem of rectangular bars is a bit more involved compared to those of elliptical and triangular bars. We shall indicate only the method of approach without going into the details. Let the sides of the rectangular cross-section be $2a$ and $2b$ with the origin at the centre, as shown in Fig. 7.9(a).

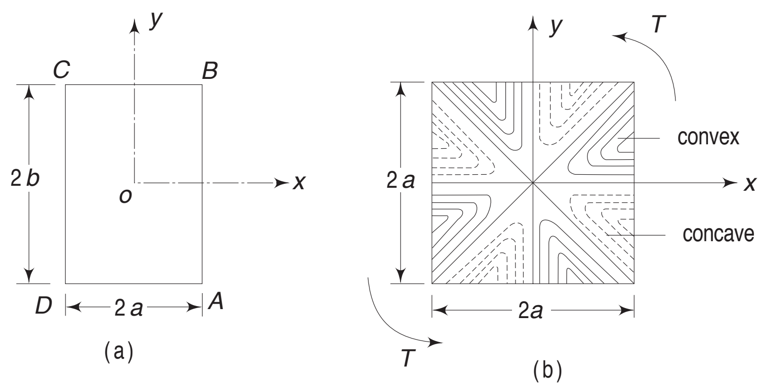

Fig. 7.9 *(a) Cross-section of a rectangular bar (b) Warping of a square section*

Our equations are, as before,

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

over the whole region R of the rectangle, and

$$\left(\frac{\partial \psi}{\partial x} - y\right) n_x + \left(\frac{\partial \psi}{\partial y} + x\right) n_y = 0$$

on the boundary. Now on the boundary lines $x = \pm a$ or AB and CD, we have $n_x = \pm 1$ and $n_y = 0$. On the boundary lines BC and AD, we have $n_x = 0$ and $n_y = \pm 1$. Hence, the boundary conditions become

$$\frac{\partial \psi}{\partial x} = y \qquad \text{on} \quad x = \pm a$$

$$\frac{\partial \psi}{\partial y} = -x \qquad \text{on} \quad y = \pm b$$

These boundary conditions can be transformed into more convenient forms if we introduce a new function ψ_1, such that

$$\psi = xy - \psi_1$$

In terms of ψ_1, the governing equation is

$$\frac{\partial^2 \psi_1}{\partial x^2} + \frac{\partial^2 \psi_1}{\partial y^2} = 0$$

over region R, and the boundary conditions become

$$\frac{\partial \psi_1}{\partial x} = 0 \qquad \text{on} \quad x = \pm a$$

$$\frac{\partial \psi_1}{\partial y} = 2x \qquad \text{on} \quad y = \pm b$$

It is assumed that the solution is expressed in the form of infinite series

$$\psi = \sum_{n=0}^{\infty} X_n(x) Y_n(y)$$

where X_n and Y_n are respectively functions of x alone and y alone. Substitution into the Laplace equation for ψ_1 yields two linear ordinary differential equations with constant coefficients. Further details of the solution can be obtained by referring to books on theory of elasticity. The final results which are important are as follows:

The function J is given by

$$J = Ka^3 b$$

For various b/a ratios, the corresponding values of K are given in Table 7.1. Assuming that $b > a$, it is shown in the detailed analysis that the maximum

Table 7.1

b/a	K	K_1	K_2
1	2.250	1.350	0.600
1.2	2.656	1.518	0.571
1.5	3.136	1.696	0.541
2.0	3.664	1.860	0.508
2.5	3.984	1.936	0.484
3.0	4.208	1.970	0.468
4.0	4.496	1.994	0.443
5.0	4.656	1.998	0.430
10.0	4.992	2.000	0.401
∞	5.328	2.000	0.375

shearing stress is at the mid-points of the long sides $x = \pm a$ of the rectangle. On these sides

$$\tau_{zx} = 0 \qquad \text{and} \qquad \tau_{\max} = K_1 \frac{Ta}{J}$$

The values of K_1 for various values of b/a are given in Table 7.1. Substituting for J, the above expression can be written as

$$\tau_{\max} = K_2 \frac{Ta}{a^2 b}$$

where K_2 is another numerical factor, as given in Table 7.1. For a square section, i.e. $b/a = 1$, the warping is as shown in Fig. 7.9 (b). The zones where u_z is positive are shown by solid lines and the zones where u_z is negative are shown by dotted lines.

Empirical Formula for Squatty Sections

Equation (7.32), which is applicable to an elliptical section, can be written as

$$\frac{T}{\theta} = \frac{\pi a^3 b^3}{a^2 + b^2} G = \frac{1}{4\pi^2} \frac{GA^4}{I_p}$$

where $A = \pi ab$ is the area of the ellipse, and $I_p = \frac{(a^2 + b^2)}{4} A$ is the polar moment of inertia. This formula is applicable to a large number of squatty sections with an error not exceeding 10%. If $4\pi^2$ is replaced by 40, the mean error becomes less than 8% for many sections. Hence,

$$\frac{T}{\theta} = \frac{GA^4}{40 I_p}$$

is an approximate formula that can be applied to many sections other than elongated or narrow sections (see Secs 7.10 and 7.11).

7.7 MEMBRANE ANALOGY

From the examples worked out in the previous sections, it becomes evident that for bars with more complicated cross-sectional shapes, analytical solutions tend to become more involved and difficult. In such situations, it is desirable to resort to other techniques—experimental or otherwise. The membrane analogy introduced by Prandtl has proved very valuable in this regard. Let a thin homogeneous membrane like a thin rubber sheet be stretched with uniform tension and fixed at its edge, which is a given curve (the cross-section of the shaft) in the *xy*-plane (Fig. 7.10).

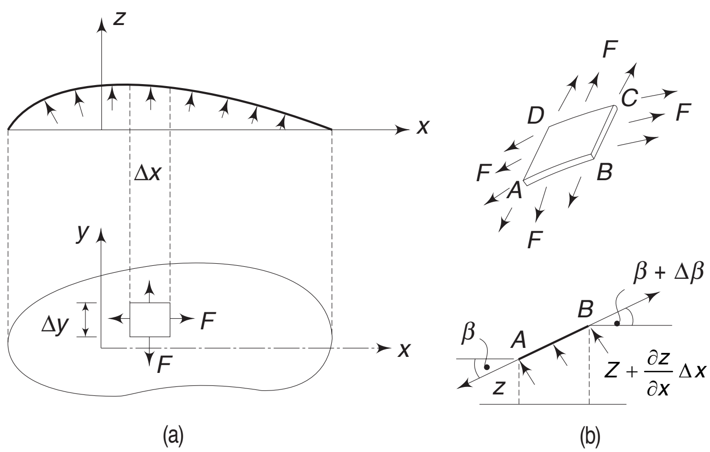

Fig. 7.10 *Stretching of a membrane*

When the membrane is subjected to a uniform lateral pressure *p*, it undergoes a small displacement *z* where *z* is a function of *x* and *y*. Consider the equilibrium of an infinitesimal element *ABCD* of the membrane after deformation. Let *F* be the uniform tension per unit length of the membrane. The value of the initial tension *F* is large enough to ignore its change when the membrane is blown up by the small pressure *p*. On face *AD*, the force acting is $F\Delta y$. This is inclined at an angle β to the *x*-axis. tan β is the slope of the face *AB* and is equal to $\partial z/\partial x$. Hence, the component of $F\Delta y$ in *z* direction is $\left(-F\Delta y \dfrac{\partial z}{\partial x}\right)$ since sin β ≈ tan ≈ β for small values of β. The force on face *BC* is also $F\,\Delta y$ but is inclined at an angle $(\beta + \Delta\beta)$ to the *x*-axis. Its slope is therefore

$$\frac{\partial z}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\Delta x$$

and the component of the force in *z* direction is

$$F\Delta y\left[\frac{\partial z}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\Delta x\right]$$

Similarly, the components of the forces $F\Delta y$ acting on faces *AB* and *CD* are

$$-F\Delta x\frac{\partial z}{\partial y} \quad \text{and} \quad F\Delta x\left[\frac{\partial z}{\partial y} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\Delta y\right]$$

Therefore, the resultant force in z direction due to tension F is

$$-F\,\Delta y\frac{\partial z}{\partial x}+F\,\Delta y\left[\frac{\partial z}{\partial x}+\frac{\partial^2 z}{\partial x^2}\Delta x\right]-F\Delta x\frac{\partial z}{\partial y}$$

$$+\,F\,\Delta x\left[\frac{\partial z}{\partial y}+\frac{\partial^2 z}{\partial y^2}\Delta y\right]$$

$$=F\left(\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}\right)\Delta x\,\Delta y$$

The force p acting upward on the membrane element $ABCD$ is $p\,\Delta x\,\Delta y$, assuming that the membrane deflection is small. For equilibrium, therefore

$$F\left(\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}\right)=-p$$

or

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=-\frac{p}{F} \tag{7.43}$$

Now, if we adjust the membrane tension F or the air pressure p such that p/F becomes numerically equal to $2G\theta$, then Eq. (7.43) of the membrane becomes identical to Eq. (7.25) of the torsion stress function ϕ. Further, if the membrane height z remains zero at the boundary contour of the section, then the height z of the membrane becomes numerically equal to the torsion stress function [Eq. (7.23)]. The slopes of the membrane are then equal to the shear stresses and these are in a direction perpendicular to that of the slope. The twisting moment is numerically equivalent to twice the volume under the membrane [Eq. (7.24)].

7.8 TORSION OF THIN-WALLED TUBES

Consider a thin-walled tube subjected to torsion. The thickness of the tube need not be uniform (Fig. 7.11). Since the thickness is small and the boundaries are free, the shear stresses will be essentially parallel to the boundary. Let τ be the magnitude of the shear stress and t the thickness.

Consider the equilibrium of an element of length Δl, as shown. The areas of cut faces AB and CD are respectively $t_1\,\Delta l$ and $t_2\,\Delta l$. The shear stresses (complementary shears) are τ_1 and τ_2. For equilibrium in z direction we should have

$$-\tau_1 t_1\;\Delta l+\tau_2\,t_2\;\Delta l=0$$

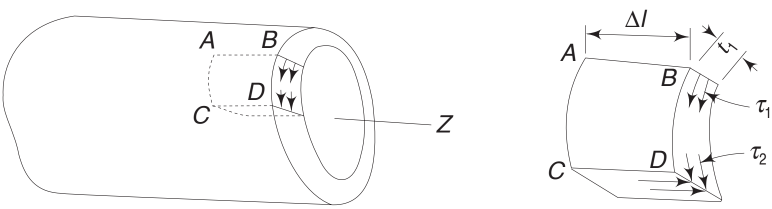

Fig. 7.11 *Torsion of a thin-walled tube*

or $$\tau_1 t_1 = \tau_2 t_2 = q, \quad \text{a constant} \tag{7.44}$$

Hence, the quantity τt is a constant. This is called the shear flow q, since the equation is similar to the flow of an incompressible liquid in a tube of varying area. For continuity, we should have $V_1 A_1 = V_2 A_2$, where A is the area and V the corresponding velocity of the fluid there.

Consider next the torque of the shear about point O [Fig. 7.12(a)].

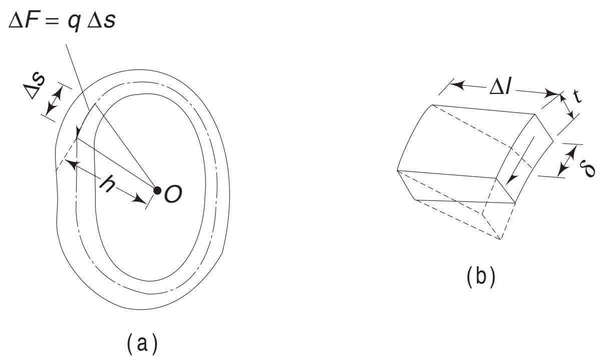

Fig. 7.12 *Cross-section of a thin-walled tube and torque due to shear*

The force acting on an elementary length Δs of the tube is

$$\Delta F = \tau t \, \Delta s = q \, \Delta s$$

The moment arm about O is h and hence, the torque is

$$\Delta T = q \, \Delta s h = 2q \, \Delta A$$

where ΔA is the area of the triangle enclosed at O by the base s. Hence, the total torque is

$$T = \Sigma \, 2q \, \Delta A = 2qA \tag{7.45}$$

Where A is the area enclosed by the centre line of the tube. Equation (7.45) is generally known as the Bredt–Batho formula.

To determine the twist of the tube, we make use of Castigliano's theorem. Referring to Fig. 7.12(b), the shear force on the element is $\tau t \, \Delta s = q \, \Delta s$. Because of shear strain γ, the force does work equal to

$$\Delta U = \frac{1}{2} (\tau t \, \Delta s) \, \delta$$

$$= \frac{1}{2} (\tau t \, \Delta s) \, \gamma \Delta l$$

$$= \frac{1}{2} (\tau t \, \Delta s) \, \Delta l \, \frac{\tau}{G}$$

$$= \frac{q^2 \, \Delta l}{2G} \frac{\Delta s}{t} \tag{7.46}$$

$$= \frac{T^2 \, \Delta l}{8A^2 G} \frac{\Delta s}{t} \tag{7.47}$$